Effect of interaction/nonlinearity on the dynamics in Anderson insulators: quantum versus classical

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Thanks to: I. Aleiner, B. Altshuler, S. Flach, O. Yevtushenko

Outline

- Introduction: Anderson localization and its manifestations

- Many-body localization for interacting electrons in disorder

- Transport due to chaos in a classical nonlinear disordered system

Anderson localization in linear/non-interacting systems

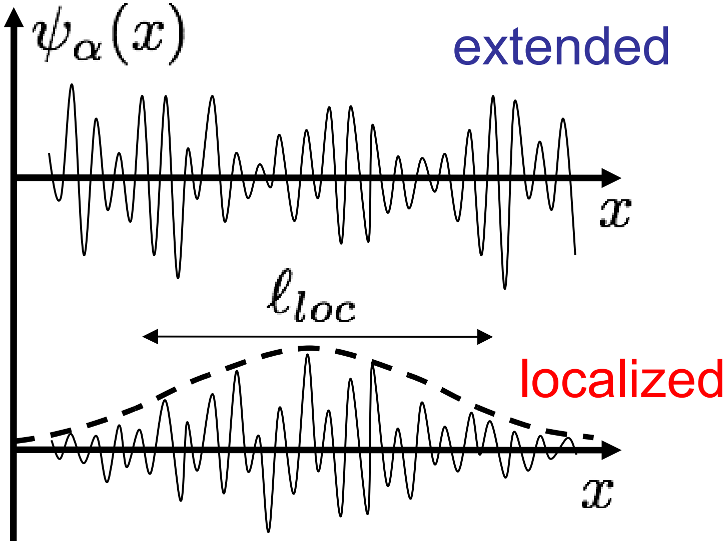
> *All happy families are alike;*
> *each unhappy family is unhappy in its own way.*
>
> L. N. Tolstoy, *Anna Karenina*

Eigenfunctions of a linear differential operator with random coefficients

$$\epsilon\psi_n + J(\psi_{n+1} + \psi_{n-1}) - v_n\psi_n = 0$$

$$\left[\epsilon + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - v(x)\right]\psi(x) = 0$$

$$\left[\frac{\omega^2}{c^2(x)} + \frac{\partial^2}{\partial x^2}\right]\psi(x) = 0$$

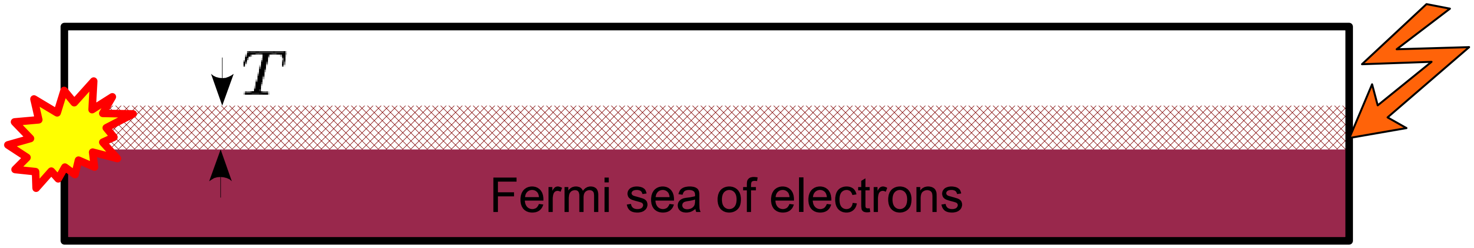


In one dimension all eigenstates are localized

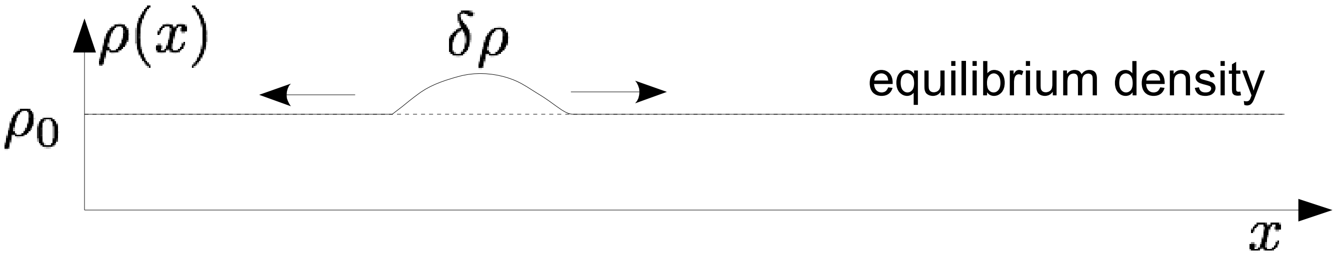
Gertsenshtein
& Vasil'ev, 1959

Manifestations in transport

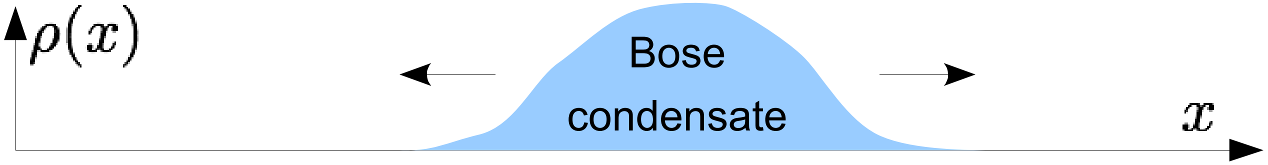
1. Absence of remote response to a local perturbation



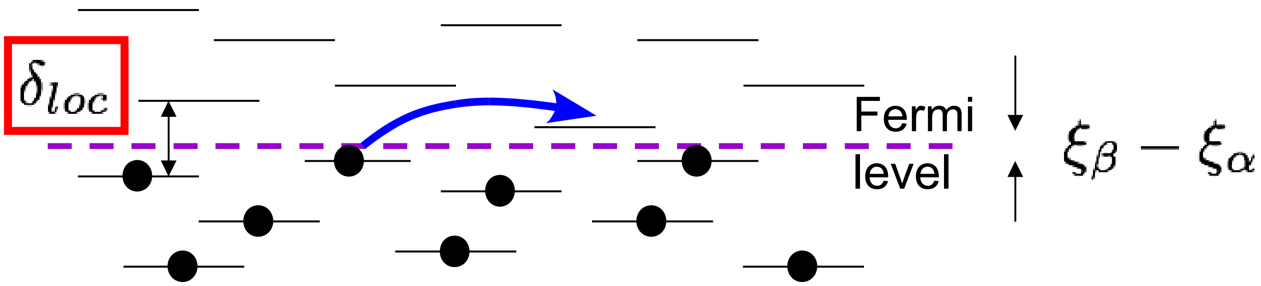
2. Absence of spreading of a small excess density



3. Absence of spreading of an initially localized cloud



Inelastic processes → transitions between localized states



δ_{loc}

Fermi level

$\xi_\beta - \xi_\alpha$

energy mismatch

Level spacing in the localization volume: $\delta_{loc} = \dfrac{1}{\nu \ell_{loc}^{d}}$

DoS per unit volume

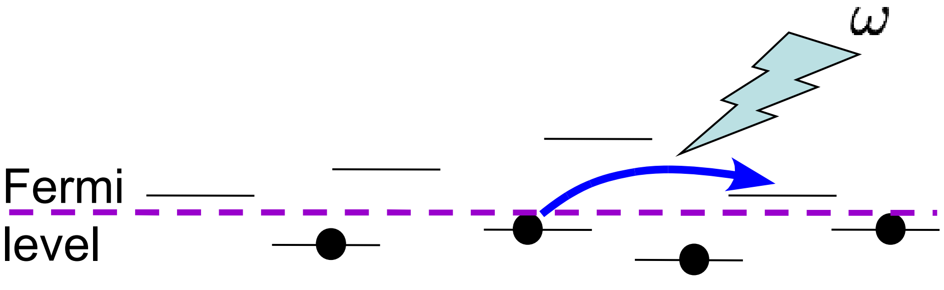
localization volume

The dc conductivity $\sigma(T)$ is determined by the inelastic rate

$T = 0 \quad \Rightarrow \quad \sigma = 0$ (any mechanism)

$\boxed{T \to 0 \quad \Rightarrow \quad \sigma = ?}$

Phonon-assisted hopping



$$\omega = \xi_\beta - \xi_\alpha \ll \delta_{loc}$$

energy difference can always be matched by a phonon

Mott formula: $\sigma(T) \propto T^\gamma \exp\left[-\left(\frac{\delta_{loc}}{T}\right)^{1/(d+1)}\right]$

1. Without Coulomb gap
2. Wrong in d=1

mechanism-dependent prefactor

any bath with a continuous spectrum of delocalized excitations down to ***E*= *0***

No phonons, **e-e** interaction $\begin{cases}\text{weak}\\ \text{short-range}\end{cases}$

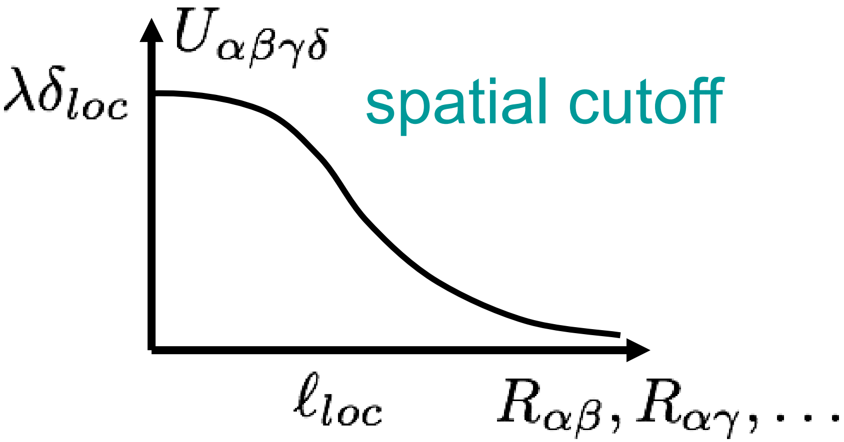
dimensionless interaction strength $\lambda \ll 1$

$$U(\mathbf{r}-\mathbf{r}') = \frac{\lambda}{\nu}\delta(\mathbf{r}-\mathbf{r}')$$

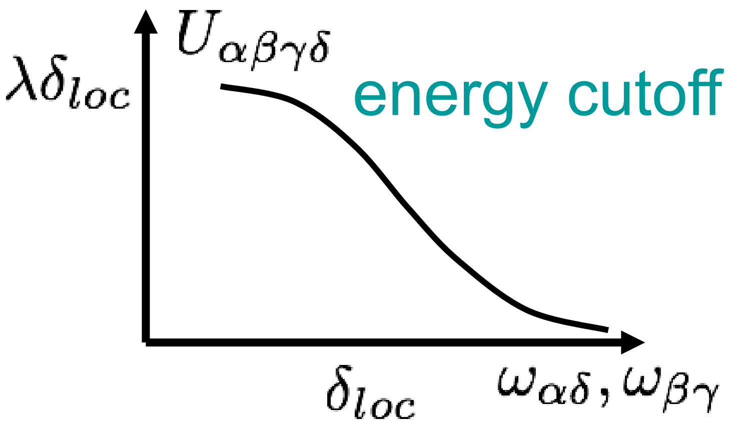
interaction range $\ll \ell_{loc}$

DoS per unit volume

Matrix elements between localized wave functions:



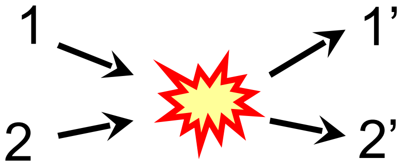
spatial cutoff



energy cutoff

Energy conservation problem

emission of electron-hole pair ↔ two-electron "collision":



$$|\xi_1 + \xi_2 - \xi_{1'} - \xi_{2'}| \sim \lambda\delta_{loc}$$

$\frac{T}{\delta_{loc}}$ – number of electrons to choose from

$$\boxed{\frac{T}{\delta_{loc}} \sim \frac{1}{\lambda}}$$

triple "collision":

$$|\xi_1 + \xi_2 + \xi_3 - \xi_{1'} - \xi_{2'} - \xi_{3'}| \sim \frac{(\lambda\delta_{loc})^2}{\delta_{loc}}$$

1. Lower temperatures → harder to conserve energy
2. Need to consider **ALL** many-electron processes to **ALL** orders of perturbation theory

Anderson localization in the many-body Fock space

(Altshuler, Gefen,
Kamenev, Levitov, 1997)

$$\begin{aligned} \xi_\alpha \quad &\rightarrow \quad \xi_\gamma + \xi_\delta - \xi_\beta \rightarrow \\ &\rightarrow \quad \xi_1 + \xi_2 + \xi_3 - \xi_4 - \xi_5 \rightarrow \ldots \end{aligned}$$

many-body Fock states → sites with random energies

e-e interaction → coupling between sites

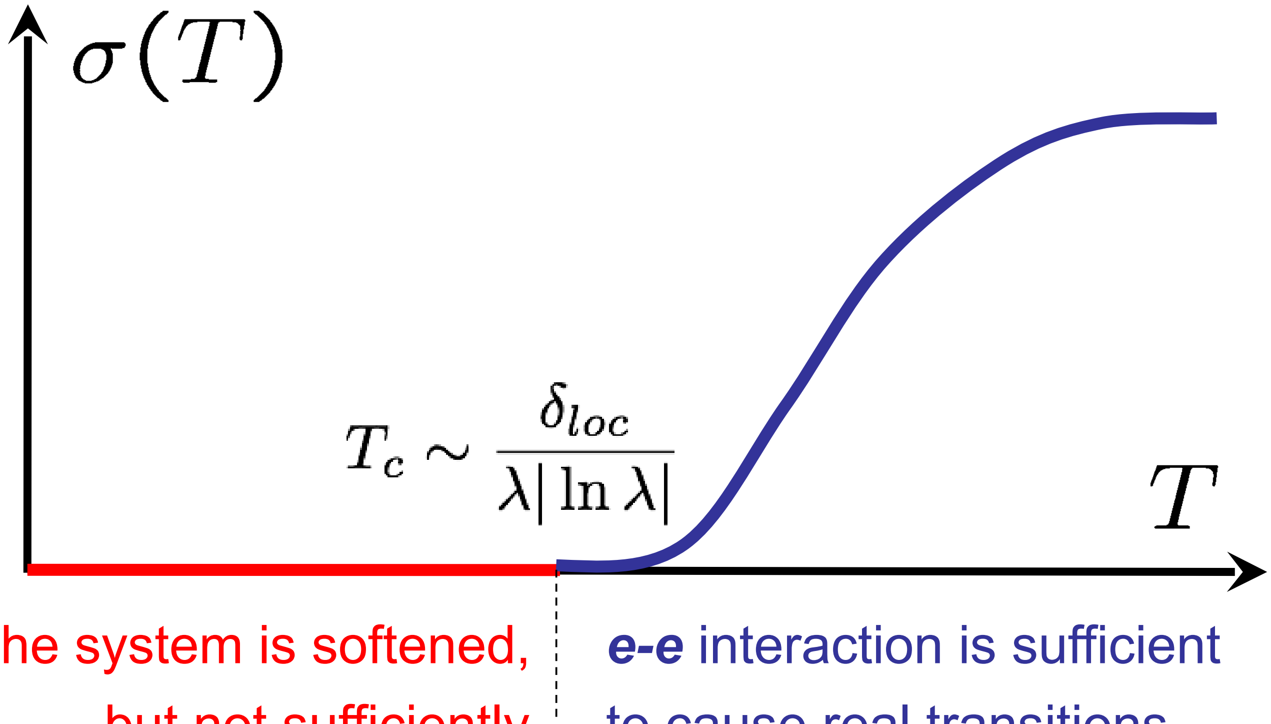
metal-insulator transition → Anderson transition

temperature → coordination number

Systematic treatment of many-electron transitions:

D. B., I. Aleiner, and B. Altshuler, Ann. Phys. **321**, 1126–1205 (2006)

The answer is $\sigma(T) = 0,\ T < T_c$



the system is softened,
but not sufficiently

e-e interaction is sufficient
to cause real transitions

NO RELAXATION

finite-temperature metal-insulator transition

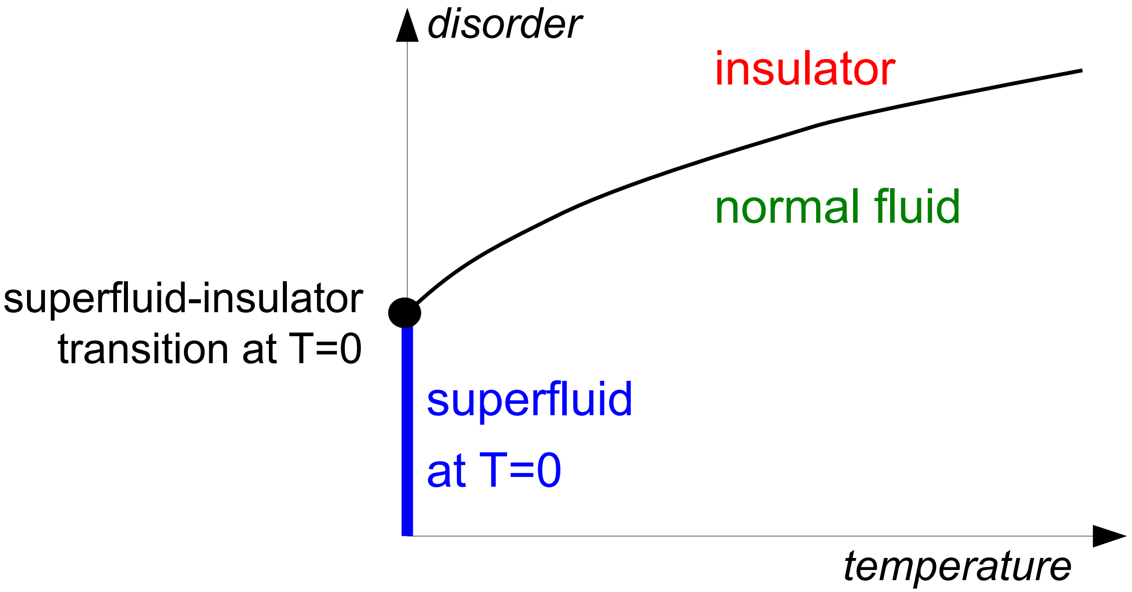
D. B., I. Aleiner, and B. Altshuler, Ann. Phys. **321**, 1126–1205 (2006)

nature physics

ARTICLES

A finite-temperature phase transition for disordered weakly interacting bosons in one dimension

I. L. Aleiner[1]*, B. L. Altshuler[1] and G. V. Shlyapnikov[2,3]



Discrete nonlinear Schrödinger equation with disorder

$$i\,\frac{d\psi_n}{dt} = \underbrace{\omega_n\psi_n - \Omega(\psi_{n+1}+\psi_{n-1})}_{\text{Anderson localization}} + \underset{\text{nonlinearity}}{g|\psi_n|^2\psi_n}$$

Classical limit of the bosonic problem: $\hbar \to 0,\ N \to \infty,\ N\hbar = \mathrm{const}$

Discrete nonlinear Schrödinger equation with disorder

$$i\frac{d\psi_n}{dt} = \underbrace{\omega_n\psi_n - \Omega(\psi_{n+1}+\psi_{n-1})}_{\text{Anderson localization}} + g|\psi_n|^2\psi_n$$

nonlinearity

Classical limit of the bosonic problem: $\hbar \to 0,\ N \to \infty,\ N\hbar = \mathrm{const}$

$$H(p,q) = \sum_n \left(\frac{p_n^2}{2m} + \frac{m\omega_n^2}{2}q_n^2\right) + \frac{g}{2}\sum_n \left(\frac{p_n^2}{2m\omega_n} + \frac{m\omega_n}{2}q_n^2\right)^2$$

anharmonic oscillators

$$-\Omega\sum_n \left(\frac{p_np_{n+1}}{2m\sqrt{\omega_n\omega_{n+1}}} + \frac{m\sqrt{\omega_n\omega_{n+1}}}{2}q_nq_{n+1}\right)$$

nearest-neighbor coupling

Hamilton's equations of motion

$$\frac{dq_n}{dt} = \frac{\partial H}{\partial p_n}, \quad \frac{dp_n}{dt} = -\frac{\partial H}{\partial q_n}$$

change of variables

$$\psi_n = \frac{p_n}{\sqrt{2m\omega_n}} + \sqrt{\frac{m\omega_n}{2}}\,iq_n$$

dimensionality: $|\psi|^2$ = action

nonlinear Schrödinger equation

DNLSE and related systems

Stationary solutions of DNLSE: Iomin, Fishman (2007); Fishman *et al.* (2008); Bodyfelt *et al.* (2010)

Transmission of a finite sample: Gredeskul, Kivshar (1992); Tietsche, Pikovski (2008)

Existence of the superfluid flow: Paul *et al*. (2005,2007,2009); Albert *et al.* (2008,2010)

Wave packet spreading in discrete DNLSE: Shepelyansky (1993); Molina, Tsironis (1994); Molina (1998); Kopidakis *et al.* (2008); Pikovsky, Shepelyansky (2008); Fishman *et al.* (2008); Bourgain, Wang (2008); Wang, Zhang (2009); Flach *et al.* (2009); Skokos *et al.* (2009); Fishman *et al.* (2009); Veksler *et al.* (2009); Krivolapov *et al.* (2009); Veksler *et al.* (2010); Iomin (2010); Skokos, Flach (2010); Flach (2010); Mulansky, Pikovsky (2010); Laptyeva *et al.* (2010)

Wave packet spreading in other nonlinear disordered 1D systems: Fröhlich *et al.* (1986); Kopidakis *et al.* (2008); Flach *et al.* (2009); Skokos *et al.* (2009); Garcia-Mata, Shepelyansky (2009); Krimer *et al.* (2009); Flach (2010); Laptyeva *et al.* (2010)

Thermalization in DNLSE and other nonlinear disordered 1D systems: Dhar, Lebowitz (2008); Dhar, Saito (2008); Oganesyan *et al.* (2009); Mulansky *et al.* (2009); Pikovsky, Fishman (2010)

Numerical integration: wave packet spreads as a power law in time

Wang, Zhang + Fishman *et al.*: slower than any power law

Given

1. Strong localization
2. Weak nonlinearity

$\left.\right\}$ → worst conditions for transport

3. Arbitrary initial condition with extensive norm and energy

Question: will the system equilibrate at long distances, and how?

Answer: yes, by normal nonlinear diffusion:

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} D(\rho) \frac{\partial \rho}{\partial x}$$

Mechanism: CHAOS

- concentrated on rare local chaotic spots
  (as seen by Oganesyan, Pal, Huse, 2009)
- which migrate along the chain
- and redistribute the energy between oscillators

Assumptions

$$i\,\frac{d\psi_n}{dt} = \omega_n\psi_n - \Omega(\psi_{n+1} + \psi_{n-1}) + g\psi_n^*\psi_n^2$$

disorder

$$-\frac{\Delta}{2} \le \omega_n \le \frac{\Delta}{2}$$

“tunnelling”

nonlinearity

$$g > 0$$

1. Strong localization: $\frac{\Omega}{\Delta} \equiv \tau \ll 1$

assumption about the Hamiltonian

2. Weak nonlinearity: $\frac{g|\psi_n|^2}{\Delta} \sim \rho \ll 1$

(nonlinear frequency shift << disorder)

note the invariance under $\psi_n \to C\psi_n, \quad g \to C^{-2}g$

3. Simple thermodynamics: $H < 0$

$$|H| \ll \Delta\sum_n |\psi_n|^2$$

(all oscillators are excited more or less equally; thermodynamic relations have a simple form)

assumptions about the initial conditions

Results

Two conserved quantities: total energy H, total action $I = \sum |\psi_n|^2$

Local thermalization in a **finite** time → $\mathcal{P}(\{\psi_n\}) \propto e^{-\beta(H-\mu I)}, \qquad \beta \equiv 1/T$

Global equilibration: transport of the conserved quantities

Macroscopic action density: $\rho(x) = \frac{1}{L^*} \sum_{n=x-L^*/2}^{x+L^*/2} \frac{g|\psi_n|^2}{\Delta} \approx \frac{gT}{|\mu|\Delta}$

get rid of the energy density thanks to $|H| \ll I\Delta$

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} D(\rho) \frac{\partial \rho}{\partial x} \qquad D(\rho) \sim \exp\left(-\mathcal{C} \ln^2 \frac{1}{\tau^p \rho} \ln \frac{1}{\rho}\right)$$

$$L^* \gg \exp\left(\mathcal{C} \ln^2 \frac{1}{\tau^p \rho}\right)$$

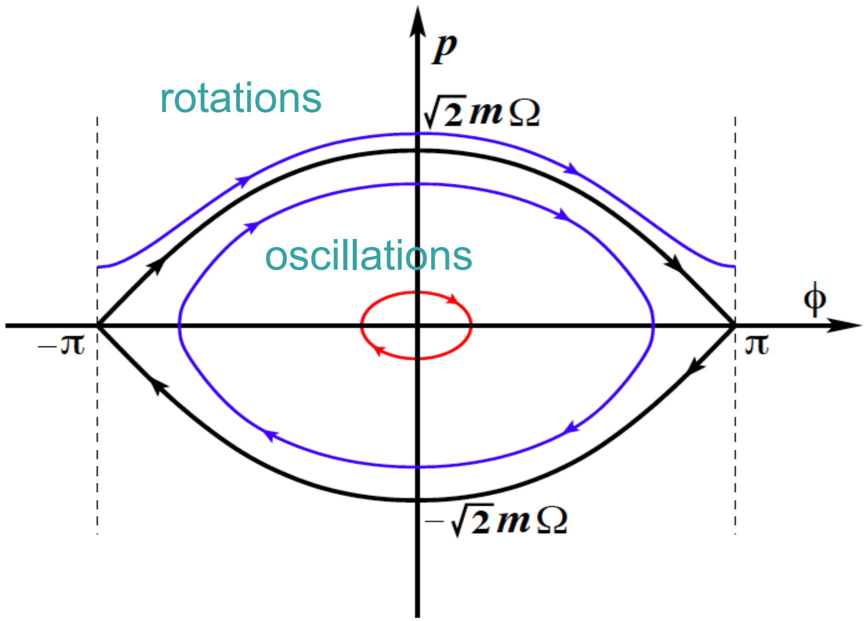
stronger than any power law

$$\frac{1}{2} \leq p \leq 3$$

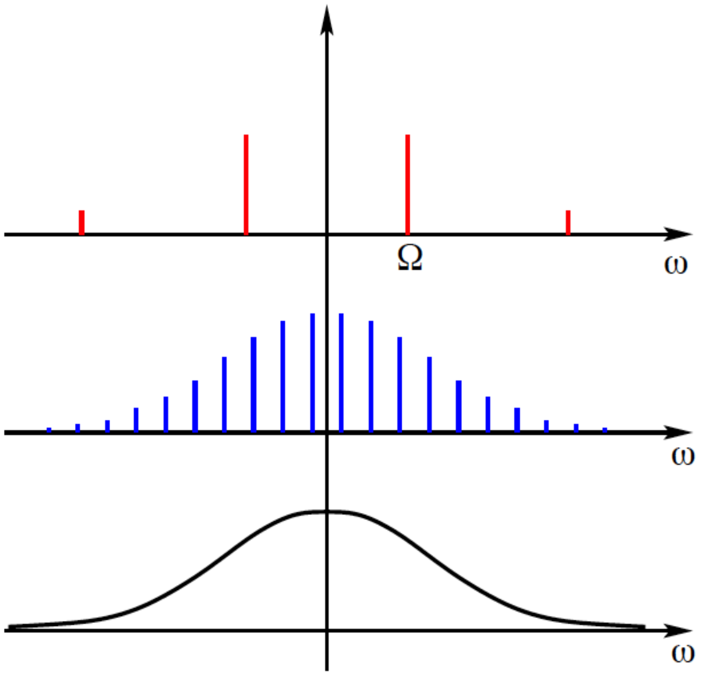
D. M. Basko, arXiv:1005.5033

Pendulum: $H(p,\phi) = \dfrac{p^2}{2m} - m\Omega^2 \cos\phi$

Phase space:



Spectrum $\int \phi(t)\, e^{i\omega t}\, dt$



the period diverges
at the separatrix

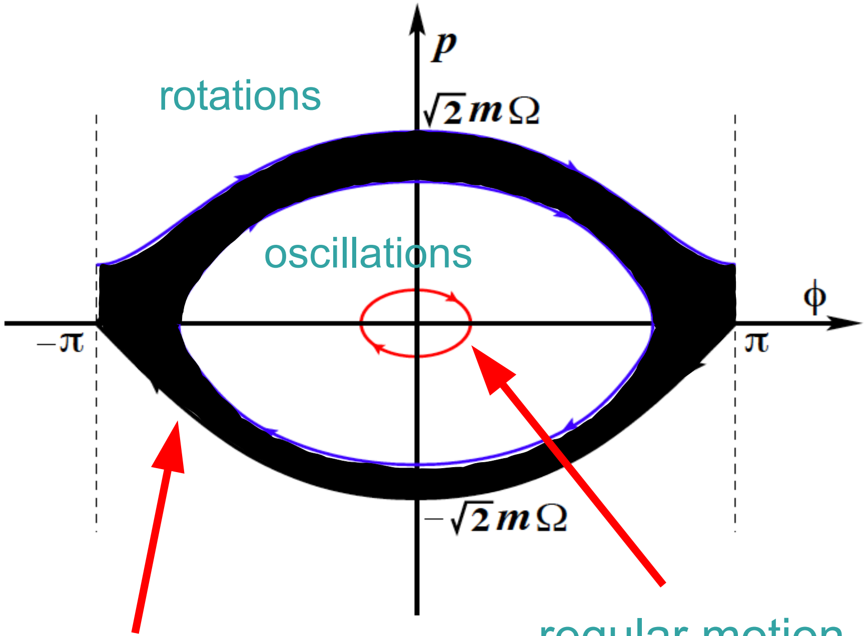
↔

the separatrix motion
has a continuous spectrum

Note: quantum pendulum has only discrete spectrum!

Perturbed pendulum:

$$H(p,\phi,t) = \frac{p^2}{2m} - m\Omega^2 \cos\phi - V\cos(\phi - \omega t)$$



rotations

oscillations

ergodic trajectories
within
the stochastic layer

regular motion
survives

Stochastic layer area:

$$W_s \equiv \int_{\text{layer}} \frac{dp\,d\phi}{2\pi} \sim \frac{V}{\Omega} e^{-|\omega|/\Omega}$$

Melnikov-Arnold integral

$$|\omega| \gg \Omega$$

Continuous spectrum
of the chaotic motion:

$$\left\langle e^{i\phi(t)} e^{-i\phi(t')} \right\rangle_\omega \sim \frac{1}{\Omega} e^{-|\omega|/\Omega}$$

review: B. Chirikov (1979)

Making a pendulum out of oscillators

action-angle variables: $\psi_n = \sqrt{I_n}\, e^{-i\phi_n}$

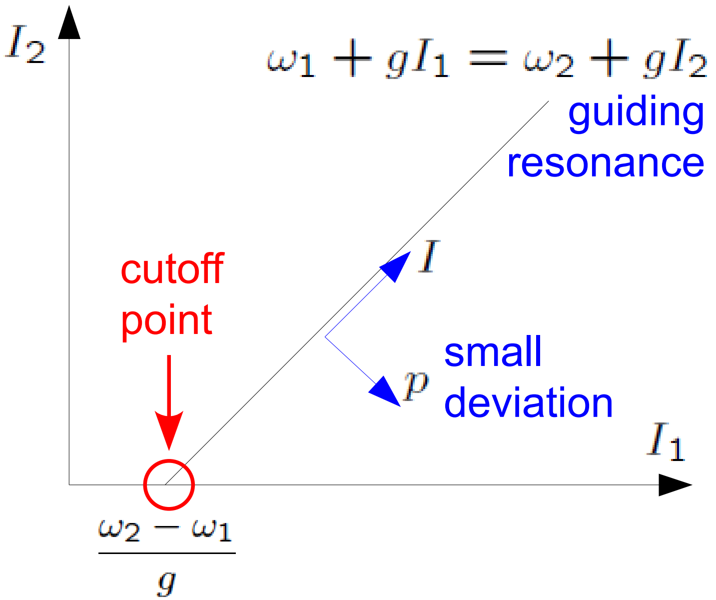
two-oscillator Hamiltonian: $$H = \boxed{\omega_1 I_1 + \frac{g I_1^2}{2}} + \boxed{\omega_2 I_2 + \frac{g I_2^2}{2}} - \boxed{2\tau\Delta\sqrt{I_1 I_2}\cos(\phi_1 - \phi_2)}$$

canonical transformation: $$I = I_1 + I_2, \quad \phi = \frac{\phi_1 + \phi_2}{2}, \quad \tilde{I} = \frac{I_1 - I_2}{2}, \quad \tilde{\phi} = \phi_1 - \phi_2$$

$$H = \underbrace{H_0(I)}_{\text{constant}} + \underbrace{(\omega_1 - \omega_2)\tilde{I} + g\tilde{I}^2}_{\text{shift}} - 2\tau\Delta\underbrace{\sqrt{I^2/4 - \tilde{I}^2}\cos\tilde{\phi}}_{\text{almost constant}}$$

$$\tilde{I} = \frac{\omega_2 - \omega_1}{2g} + p$$

$$\tau \ll 1$$



A third oscillator:

$$2\tau\Delta\sqrt{I_2 I_3}\cos(\phi_2 - \phi_3)$$

perturbation of the pendulum

Three oscillators are sufficient to generate chaos

The price of making a pendulum

To find a separatrix:

the shift is possible only if $I_1, I_2 > 0$

cutoff point

$$\omega_1 + gI_1 = \omega_2 + gI_2$$

$$\frac{H - \mu I}{T} > \underbrace{\frac{|\mu||\omega_1 - \omega_2|}{gT}}_{\sim \frac{1}{\rho}}$$

unless $|\omega_1 - \omega_2| \ll \Delta$

Look for a resonance
or
pay the thermal exponential

(guiding resonance)

To create the stochastic layer:

the pendulum frequency $\Omega \sim \sqrt{\tau \Delta g I}$

$$\frac{|\omega_2 - \omega_3|}{\Omega} \sim \frac{1}{\sqrt{\tau \rho}}$$

unless $|\omega_2 - \omega_3| \ll \Delta$

Look for another resonance
or
pay the Melnikov-Arnold exponential

(layer resonance)

Chaotic oscillators are rare:

density $\sim \min\{\tau\rho, \rho^2\}$

Making a pendulum out of more oscillators

$$-\tau\Delta(\psi_1^*\psi_2+\psi_2^*\psi_1)+\omega_2\psi_2^*\psi_2-\tau\Delta(\psi_2^*\psi_3+\psi_3^*\psi_2)$$

effective coupling 1 ↔ 3: $\frac{(\tau\Delta)^2}{\omega_1-\omega_2}(\psi_1^*\psi_3+\psi_3^*\psi_1)$

works when $\omega_1\approx\omega_3\neq\omega_2$

Tunnelling + nonlinearity → effective couplings of the form

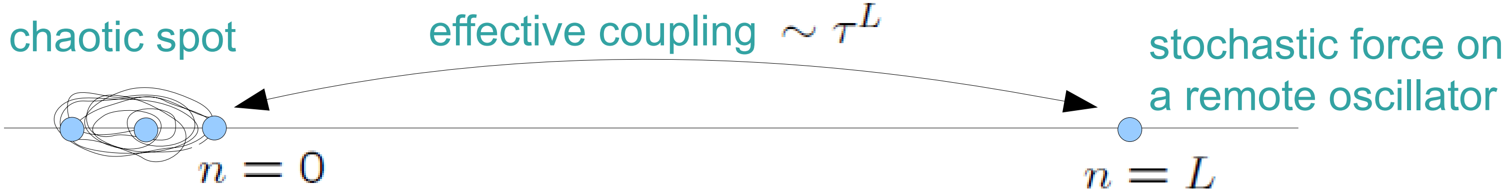
$$\psi_1^*\psi_2^*\psi_3^*\psi_4\psi_5\psi_6\rightarrow\cos(\phi_1+\phi_2+\phi_3-\phi_4-\phi_5-\phi_6)$$

Guiding and layer resonances can be generated in high orders of the perturbation theory

Competition: number of combinations ↔ power of the coupling constants

size of a chaotic spot << distance between chaotic spots

Arnold diffusion



chaotic spot

effective coupling $\sim \tau^L$

stochastic force on a remote oscillator

$n = 0$

$n = L$

Effective coupling $2V_{m_1 \ldots m_N} \cos(m_1\phi_1 + \ldots + m_N\phi_N)$

action conservation: $m_1 + \ldots + m_N = 0$

Change in actions due to the stochastic force after time t

$$\langle \delta I_n \delta I_{n'} \rangle \sim t\, V^2_{m_1 \ldots m_N} \frac{m_n m_{n'}}{\Omega} \exp\left(- \frac{|m_1\omega_1 + \ldots + m_N\omega_N|}{\Omega} \right)$$

probability = $\underbrace{f(\{I_n\})}_{\text{distribution function}} W_s\, g|\vec{m}^{\mathrm{g}}|^2\, \delta\big(\underbrace{m_1^{\mathrm{g}}(\omega_{n_1^{\mathrm{g}}} + gI_{n_1^{\mathrm{g}}}) + \ldots + m_N^{\mathrm{g}}(\omega_{n_N^{\mathrm{g}}} + gI_{n_N^{\mathrm{g}}})}_{\text{guiding resonance}}\big) \prod_n dI_n$

stochastic layer area

$$W_s \frac{\partial f}{\partial t} = \sum_{n,n'} \frac{\partial}{\partial I_n} W_s D_{nn'} \frac{\partial f}{\partial I_{n'}}$$

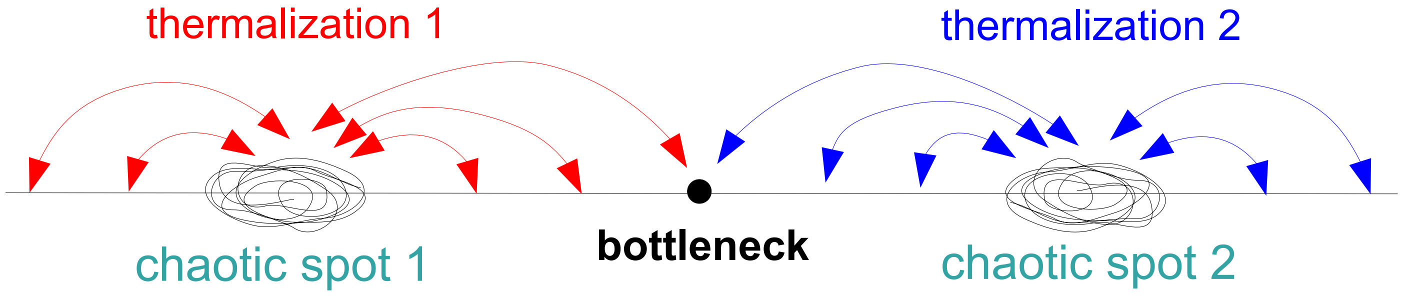
Constraints on the diffusion:

1. Total action is conserved

2. Total energy is conserved

3. The system stays on the guiding resonance

Long-distance relaxation



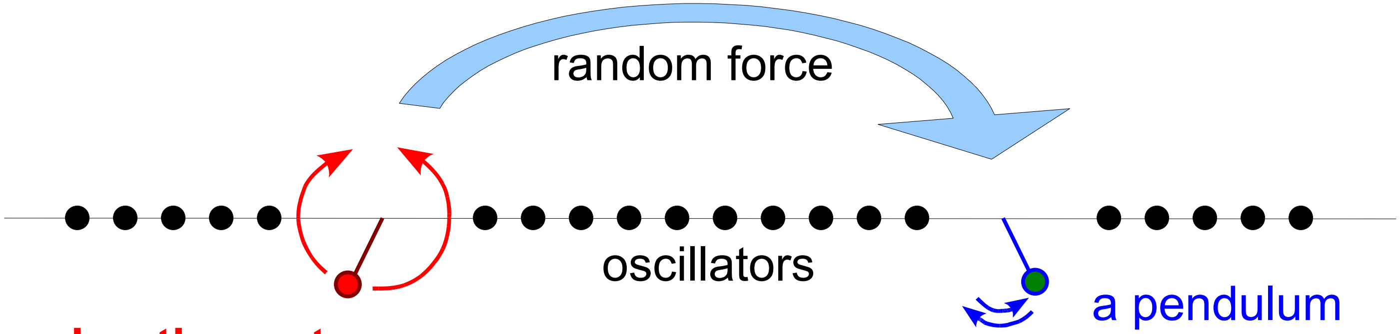
Typical density of chaotic spots $\sim \rho^2$

Coupling between the chaotic spots and the bottleneck $\sim \tau^{1/\rho^2}$

worse than activation ($\rho \propto T$)

Look for a better mechanism!

Migration of chaotic spots



chaotic spot:

a pendulum

close to the separatrix

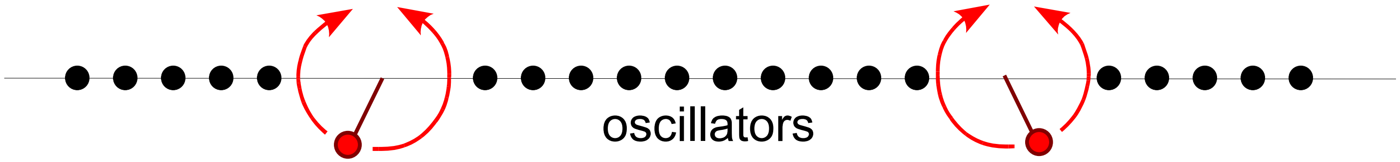
a pendulum

far from the separatrix

The pendulum cannot leave its stochastic layer

Diffusion in energy brings the pendulum into the vicinity of the separatrix (sooner or later)

Migration of chaotic spots

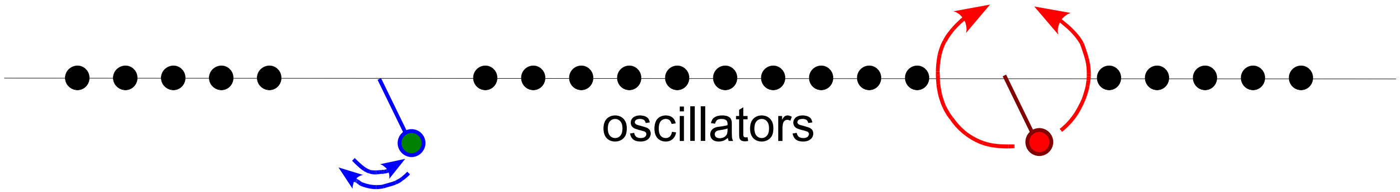


old chaotic spot

new chaotic spot

Any of the two spots can leave its stochastic layer driven by the other one

Migration of chaotic spots

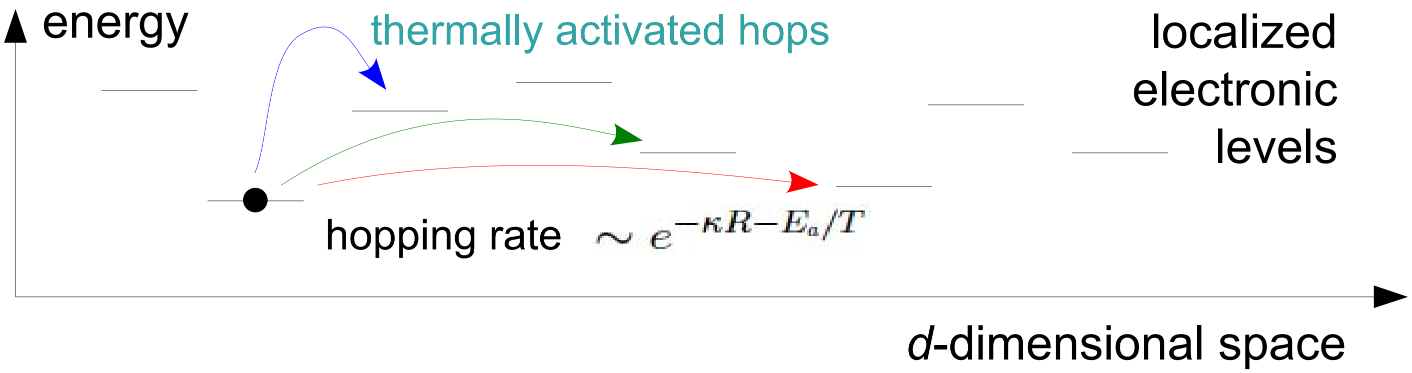


The old chaotic spot
is turned off

The new spot is now driving
the Arnold diffusion

Hopping chaotic spot does not carry energy or action

Variable-range hopping of electrons



hopping rate $\sim e^{-\kappa R - E_a/T}$

To find a low level one should explore large distances $E_a^{min} \sim \dfrac{1}{\nu R^d}$

⇩

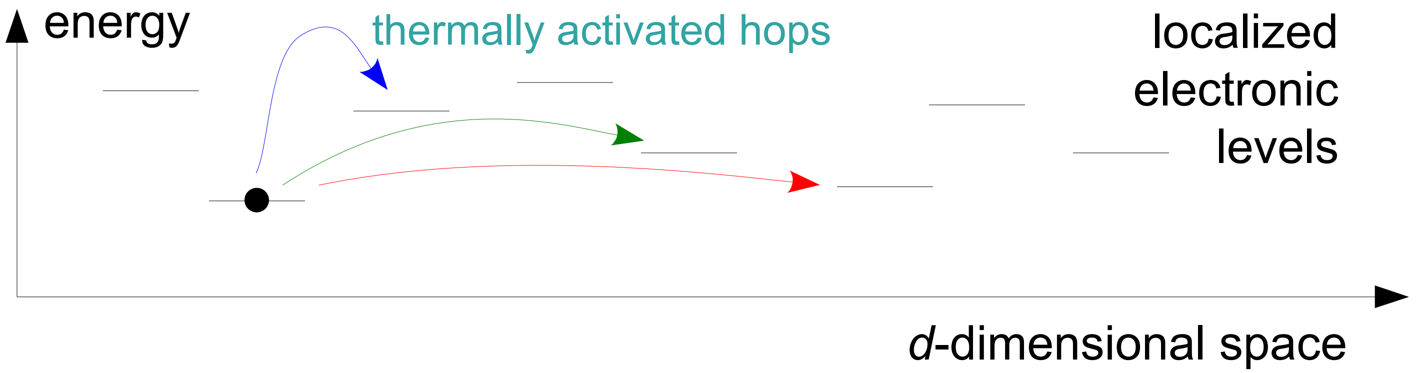
Competition between $e^{-\kappa R}$ and $e^{-E_a/T}$

$$\sigma(T) \propto \max_R e^{-\kappa R-(\nu R^d)^{-1}/T} = \exp\left[-\frac{d+1}{d}\left(\frac{\kappa^d}{\nu T}\right)^{1/(d+1)}\right]$$

Mott (1969)

stretched exponential after optimization

Variable-range hopping of electrons



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Competition between $e^{-\kappa R}$ and $e^{-E_a/T}$

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~~Mott (1969)~~

stretched exponential ~~after optimization~~

One dimension:

$$\sigma(T) \sim \exp\left(-\frac{\kappa}{2\nu T}\right)$$

Kurkijarvi (1973)

Rare “bad” regions block the transport in 1D

“Breaks”

Chaotic fraction w_n

$$w_n = \frac{1}{Z} \int_{\text{chaotic}(n)} e^{-(H-\mu I)/T} \prod_n \frac{dI_n\, d\phi_n}{2\pi}, \qquad Z \equiv \int e^{-(H-\mu I)/T} \prod_n \frac{dI_n\, d\phi_n}{2\pi}$$

all guiding resonances whose leftmost oscillator is *n*

$w = e^{-\lambda}$ for chaotic spots ⟺ $e^{-E_a/T}$ for electrons

thermal exponential

Melnikov-Arnold exponential

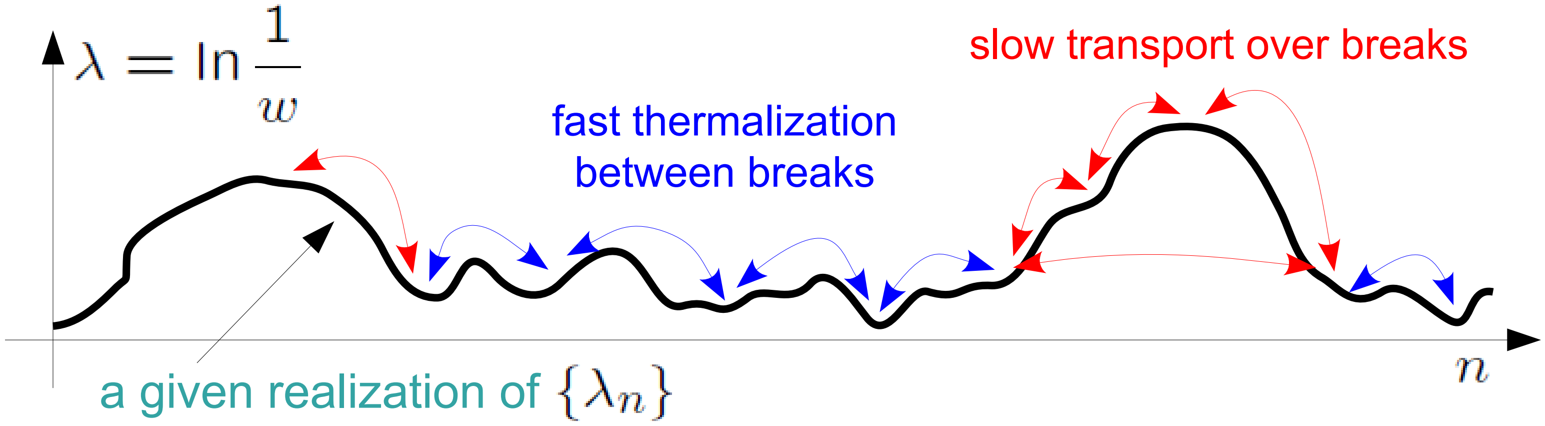
coupling constants

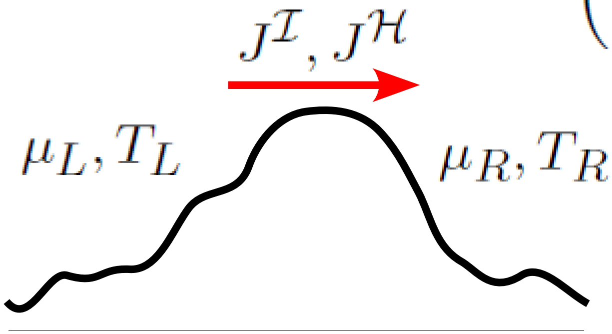
w is a random quantity determined by the disorder

Probability distribution:

$$\mathcal{P}\left\{w < w_0\right\} = \exp\left(-\mathcal{C}_1 \rho\, w_0^{1/[\mathcal{C}\ln^2(1/\tau^p\rho)]}\right)$$

From λ to σ : break resistance





1. Definition of the current

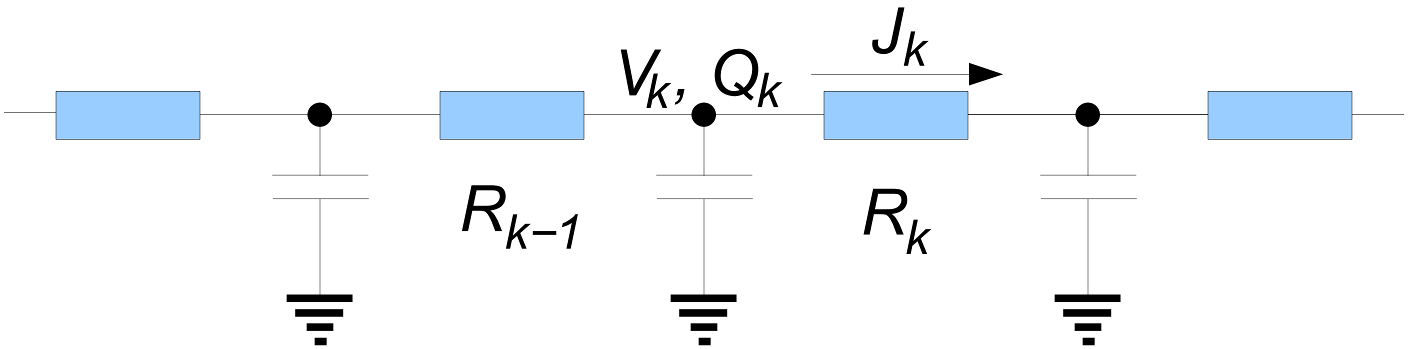
$$\begin{pmatrix} J^{\mathcal{I}} \\ J^{\mathcal{H}} \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} \langle I\rangle_R \\ \langle H\rangle_R \end{pmatrix}$$

2. The diffusion equation for actions

$$\begin{pmatrix} J^{\mathcal{I}} \\ J^{\mathcal{H}} \end{pmatrix} = R_b^{-1}\begin{pmatrix} \mu_L/T_L - \mu_R/T_R \\ 1/T_R - 1/T_L \end{pmatrix}$$

Each break can be characterized by its “resistance”

From λ to σ: resistors in series



$$\left\{\begin{aligned} \frac{dQ_k}{dt} &= J_{k-1} - J_k \\ J_k &= R_k^{-1}(V_k - V_{k+1}) \\ Q_k &= Q_k(V_k) \end{aligned}\right.$$

definition of the current

"resistance" of the break

thermodynamics

↓

$$\left\{\begin{aligned} \frac{\partial Q}{\partial t} &= \frac{\partial}{\partial x}\sigma(V)\frac{\partial V}{\partial x} \\ Q &= Q(V) \end{aligned}\right.$$

macroscopic "charge density" $Q = \frac{1}{L}\sum Q_k$

macroscopic "conductivity" $\sigma = \left(\frac{1}{L}\sum R_k\right)^{-1}$

Optimal breaks

$$\sigma^{-1} = \frac{1}{L}\sum_{\text{breaks}\in L} R_b$$

self-averaging at long distances

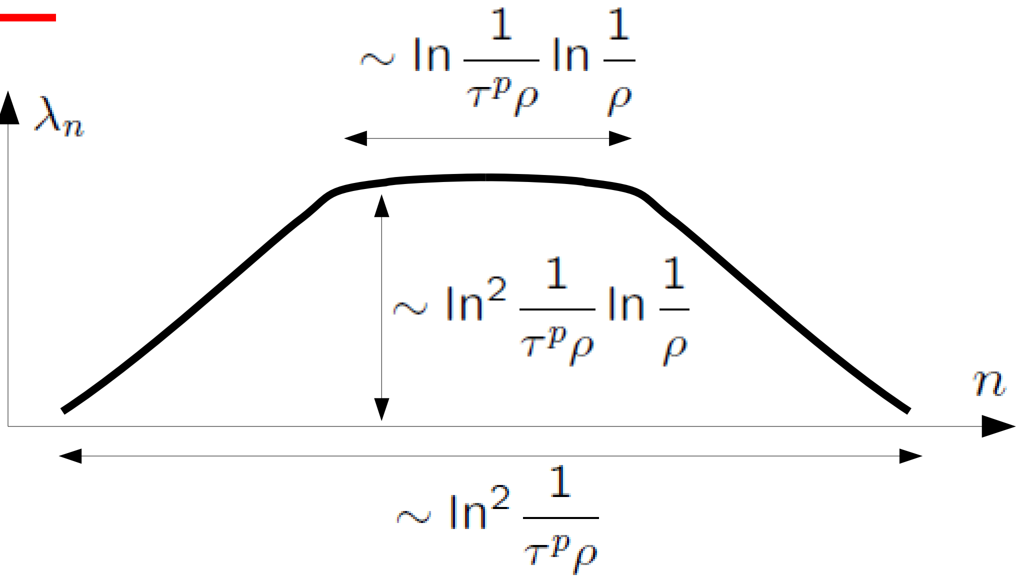
$$\sigma^{-1} = \int R_b(\{\lambda_n\})\, \underline{dP(\{\lambda_n\})}$$

probability measure
per unit length

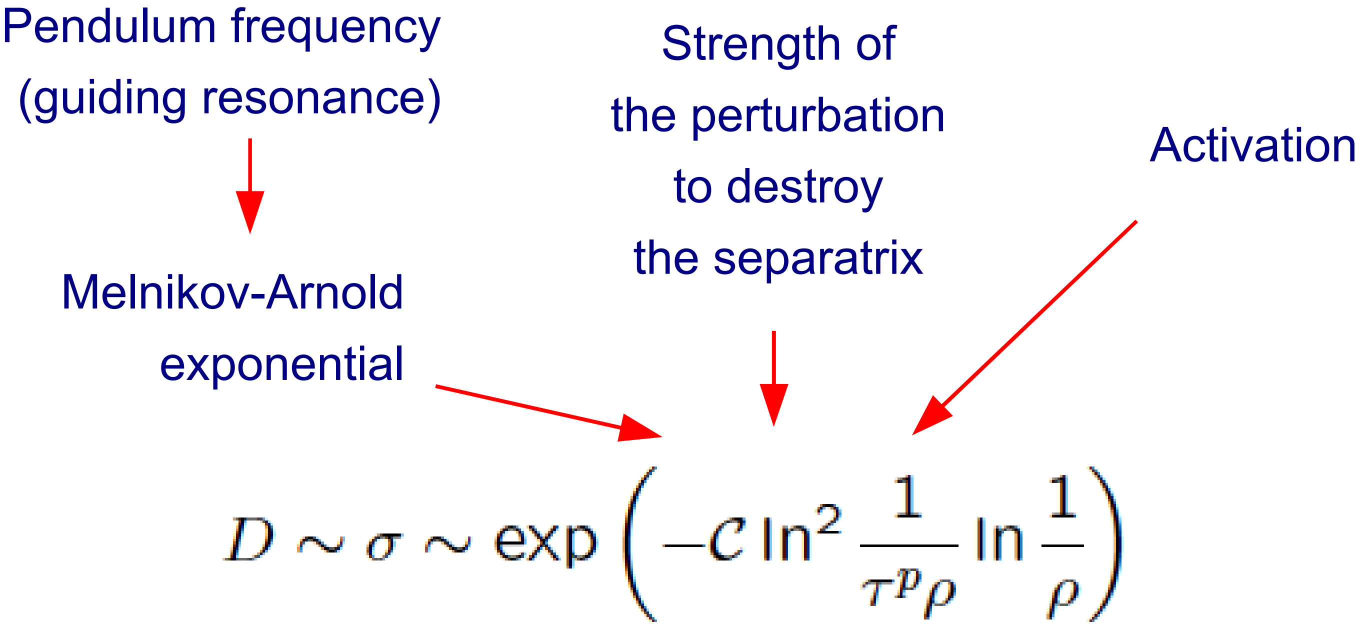
increasing

decreasing

The integral is dominated
by configurations close
the optimal one



Macroscopic diffusion coefficient: three logarithms



Pendulum frequency (guiding resonance)

Melnikov-Arnold exponential

Strength of the perturbation to destroy the separatrix

Activation

$$D \sim \sigma \sim \exp\left(-\mathcal{C}\ln^2\frac{1}{\tau^p\rho}\ln\frac{1}{\rho}\right)$$

Macroscopic length scale

$$L^* \sim \exp\left(\mathcal{C}\ln^2\frac{1}{\tau^p\rho}\right)$$

distance between the optimal breaks

Conclusions

1. In a localized interacting quantum many-body system all dynamics can be frozen. The system does not generate continuous spectrum because of many-body localization.

2. In a classical system chaotic motion can appear locally. It has continuous spectrum and drives the relaxation.