Disordered Josephson junction chains: Anderson localization of normal modes and impedance fluctuations

D. M. Basko

Laboratoire de Physique et Modélisation des Milieux Condensés

CNRS and Université Joseph Fourier Grenoble 1

Thanks to: F. Hekking, W. Guichard, V. Golovach, G. Crisan
The subject of the present talk

physical phenomenon
(its various aspects)

Anderson localization
(anisotropy of the disorder,
wave polarization,
nonlinearities, etc.)

models
Schrödinger equation,
wave equation,
tight-binding, etc.

physical systems
Electrons, light,
cold atoms, ultrasound,
Josephson junction chains

The subject of the present talk
Josephson junction

- Superconducting QUantum Interference Devices (the most sensitive magnetometers)
- Qbits
- metrology (frequency ↔ voltage)

Josephson current:

\[ I_{1\rightarrow 2} = I_c \sin(\phi_1 - \phi_2) \]

- critical current (characterizes the structure of the contact)
- superconducting phases (characterize the state of each island)

SQUID (image from W. Guichard)
RCSJ model

The total current between the islands

\[ I_{1 \rightarrow 2} = I^c \sin(\phi_1 - \phi_2) + \frac{V_1 - V_2}{R} + C \frac{d(V_1 - V_2)}{dt} \]

- **Josephson** (Cooper pairs)
- **normal** (quasiparticles)
- **displacement** (capacitance)

\[ \frac{d\phi_n}{dt} = -2eV_n \]

gauge invariance in the superconductor

**Classical regime:**

- **gigahertz** inverse time of transfer of one Cooper pair
- \[ \frac{I^c}{2e} \gg \frac{(2e)^2}{2C} \]
- charging energy due to capacitance between the islands

**femtofarads**

Otherwise: strong Coulomb blockade

(charge discreteness matters)
Josephson junction chains

V. Manucharyan et al., Science 326, 113 (2009)

- large impedance with little dissipation Maslyuk et al., PRL 109, 137002 (2012)
- control over quantum coherence (phase slips) Pop et al., Nature Phys. 6, 589 (2010)
Josephson junction chains

\[ \phi_1, \phi_2, \ldots, \phi_n, \ldots, \phi_{N-1}, \phi_N \]

\[ C_{1/2}, I_{1/2}^c, R_{1/2}, C_{1/2}^g \]

\[ V_\omega, I_\omega \]

A small capacitance between the islands and the ground

The three contributions

Nonlinear "wave" equation

Because of \( I^c \sin(\phi_n - \phi_{n+1}) \)
Small oscillations of the phase

\[
\sin(\phi_n - \phi_{n+1}) \rightarrow \phi_n - \phi_{n+1}
\]

Josephson current \(\rightarrow\) inductance

\[
Y_{n-1/2}(V_n - V_{n-1}) + Y_{n+1/2}(V_n - V_{n+1}) - i\omega C_n^q V_n = 0
\]

linear “wave” equation

\[
Y_{n+1/2}(\omega) = -\frac{1}{i\omega L_{n+1/2}} - i\omega C_{n+1/2} + \frac{1}{R_{n+1/2}}
\]

dissipation!

Non-Hermitian quadratic eigenvalue problem
No disorder, no dissipation

Infinitely long chain:  \( V_n \propto e^{i kn} \)

\[
\omega_k = \frac{1}{\sqrt{LC}} \sqrt{\frac{4 \sin^2(k/2)}{4 \sin^2(k/2) + C'g/C}}
\]

- Plasma frequency
- Linear dispersion (usual wave equation)
- Inverse screening length
No disorder, no dissipation

Infinitely long chain: \( V_n \propto e^{ikn} \)

\[
\omega_k = \frac{1}{\sqrt{LC}} \sqrt{\frac{4\sin^2(k/2)}{4\sin^2(k/2) + Cg/C}}
\]

Finite length, \( N \) junctions:

\[
k = 0, \frac{\pi}{N}, \frac{2\pi}{N}, \ldots, \frac{(N-1)\pi}{N}
\]

80 junctions:

Masluk et al., PRL 109, 137002 (2012)

- 80 junctions: experimental frequencies, theoretical calculation
- \( C_g/C \approx 10^{-3} \)
- Mode profiles
- Linear dispersion (usual wave equation)
- Inverse screening length
- Plasma frequency

\[ C_g/C \approx 10^{-3} \]
Disorder, no dissipation

Critical current, junction capacitance $\propto$ junction area – the main source of disorder

\[ L_{n+1/2} = \frac{L}{1 + \zeta_n}, \quad C_{n+1/2} = C(1 + \zeta_n), \quad \langle \zeta_n^2 \rangle = \sigma_\zeta^2 \ll 1 \]

weak relative fluctuations of the junction areas

\[ C_n^g = C^g(1 + \eta_n), \quad \langle \eta_n^2 \rangle = \sigma_g^2 \]

weak relative fluctuations of the ground capacitances
Disorder, no dissipation

Critical current, junction capacitance \( \propto \) junction area – the main source of disorder

\[
L_{n+1/2} = \frac{L}{1 + \zeta_n}, \quad C_{n+1/2} = C(1 + \zeta_n), \quad \langle \zeta_n^2 \rangle = \sigma_S^2 \ll 1
\]

weak relative fluctuations of the junction areas

\[
C_{n}^g = C^g(1 + \eta_n), \quad \langle \eta_n^2 \rangle = \sigma_g^2
\]

weak relative fluctuations of the ground capacitances

**Long chains**: the inverse localization length from the DMPK equation

\[
\frac{1}{\xi} = \frac{\sigma_S^2 + \sigma_g^2}{2} \tan^2 \frac{k}{2}
\]

at \( k \to 0 \) goes as \( k^2 \) (standard behavior for Goldstone modes)

at \( k \to \pi \) diverges

**Short chains** \( N \ll \xi \): random perturbative shifts of the discrete frequencies

\[
\langle \delta \omega_k^2 \rangle = \frac{3}{8} \frac{\sigma_S^2 + \sigma_g^2}{LC} \frac{(C^g/C)^2}{N} \frac{4 \sin^2(k/2)}{[4 \sin^2(k/2) + C^g/C]^3}
\]

motional narrowing

Basko & Hekking, PRB 88, 094507 (2013)
Disorder and dissipation

The system must be driven

Impedance of the semi-infinite chain \( Z(\omega) = \frac{V_\omega}{I_\omega} \)

Reflection coefficient of the transmission line

Statistics of reflection coefficient in the disordered Helmholtz equation with absorption:

- One mode
  - Freilikher, Pustilnik & Yurkevich, *PRL* 73, 810 (1994)
  - Pradhan & Kumar, *PRB* 50, 9644 (1994)

- Many modes
Localization and absorption

inverse penetration depth ≡ spatial Lyapunov exponent $\lambda_\omega(\sigma^2, Q^{-1})$

Taylor series (weak disorder, weak absorption)

$\lambda_\omega(\sigma^2, Q^{-1}) = a(\omega) \sigma^2 + b(\omega) Q^{-1} + \ldots$

$\frac{1}{\xi(\omega)}$ \hspace{1cm} $\kappa(\omega)$

inverse localization length \hspace{1cm} inverse absorption length

without absorption \hspace{1cm} without disorder

$\kappa = \frac{1}{2Q} \frac{\omega^2 LC \sqrt{C^9}}{\sqrt{1 - \omega^2 LC - \omega^2 LC^9 / 4}}$

Both $1/\xi, \kappa \ll k$. What happens for $\kappa \gg 1/\xi$ and $\kappa \ll 1/\xi$?
Absorption by the eigenmodes

Absorption of the transmission line:

\[ \text{Re } Z(\omega) \sim \sum_m \frac{\gamma_m A_m}{(\omega - \omega_m)^2 + \gamma_m^2} \]

\[ \omega_m + i\gamma_m \text{ complex eigenvalue of the non-Hermitian problem} \]

strength of the \( m \)th eigenmode at the location of the drive

only modes within length \( \sim \xi \) effectively contribute

Mode spacing within a localization length:

\[ \delta \xi \ll \gamma \]

\[ \delta \xi \gg \gamma \]

\[ \delta \xi \approx \left( \frac{\xi}{2\pi} \frac{dk}{d\omega_k} \right)^{-1} \]

determined by disorder only

density of states without disorder

determined by absorption only

Mode broadening:

\[ \gamma \approx \frac{d\omega_k}{dk} \kappa \]

\[ \frac{\gamma}{\delta \xi} = \frac{\kappa \xi}{2\pi} \]

the main control parameter

Dispersion law without disorder
Impedance statistics

Probability distribution of the normalized resistance

\[ P(\text{Re } Z) \sim \exp \left( -\frac{\kappa \xi}{4} \frac{|Z - Z_0|^2}{|Z_0|^2} \right) \]

Absorption dominates: \( \kappa \gg \frac{1}{\xi} \)

small mesoscopic fluctuations calculated from the DMPK equation

Localization dominates: \( \kappa \ll \frac{1}{\xi} \)

universal power-law tail here obtained from numerics

Pradhan & Kumar, *PRB* 50, 9644 (1994)
Conclusions

1. Small phase oscillation in JJ chains $\rightarrow$ a wave-like system with controllable disorder and absorption

2. Relevant observable quantity: impedance $Z(\omega)$

3. Absorption dominates over localization: weak Gaussian fluctuations

4. Localization dominates over absorption: universal power-law tail in the distribution function