Chaos and transport in disordered classical nonlinear chains

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Discrete nonlinear Schrödinger equation with disorder

$$i\frac{d\psi_n}{dt} = \omega_n\psi_n - \Omega(\psi_{n+1} + \psi_{n-1}) + g\psi_n^*\psi_n^2$$

harmonic oscillators

nearest-neighbor coupling

anharmonicity

Coordinate and momentum: $\psi_n = \frac{p_n}{\sqrt{2m\omega_n}} + \sqrt{\frac{m\omega_n}{2}}\, iq_n$

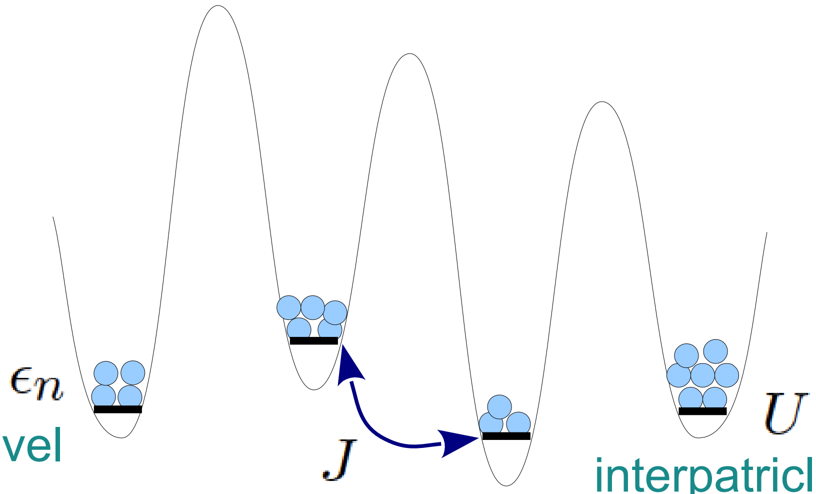
Classical equations of motion: $\frac{dp_n}{dt} = -\frac{\partial H(p.q)}{\partial q_n}, \quad \frac{dq_n}{dt} = \frac{\partial H(p,q)}{\partial p_n}$

Dimensionality: $\psi_n \sim \sqrt{\text{action}}$

Bosonic atoms in disordered optical lattices

Bose-Hubbard Hamiltonian:

$$\hat{H} = \sum_n \epsilon_n \hat{b}_n^\dagger \hat{b}_n + U \sum_n \hat{b}_n^\dagger \hat{b}_n^\dagger \hat{b}_n \hat{b}_n - J \sum_n \left(\hat{b}_{n+1}^\dagger \hat{b}_n + \hat{b}_n^\dagger \hat{b}_{n+1} \right)$$



level position

tunneling amplitude

interpatricle repulsion

bosonic operator

$$\hat{b}_n = \frac{\psi_n}{\sqrt{\hbar}} + \hat{\xi}_n$$

quantum fluctuation

condensate wave function

Billy *et al.*, *Nature* **453**, 891 (2008)

Roati *et al.*, *Nature* **453**, 895 (2008)

$$\frac{d\hat{b}_n}{dt} = \frac{i}{\hbar} [\hat{H}, \hat{b}_n] \quad \Rightarrow \quad i\hbar \frac{d\psi_n}{dt} = \epsilon_n \psi_n - J(\psi_{n-1} + \psi_{n+1}) + \frac{U}{\hbar} |\psi_n|^2 \psi_n$$

The classical limit of a bosonic field: $N_n \equiv \langle \hat{b}_n^\dagger \hat{b}_n \rangle \rightarrow \infty, \; \hbar \rightarrow 0, \; N_n \hbar \rightarrow |\psi_n|^2$

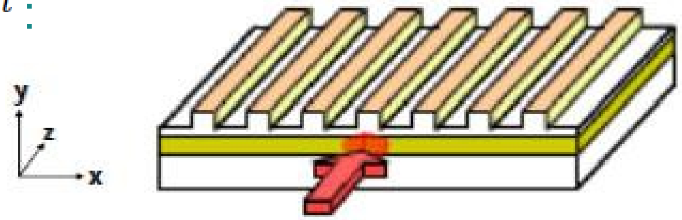
Light in evanescently coupled 1D waveguides

Wave equation for the electric field $E(\mathbf{r})e^{-i\omega t}$:

$$\varepsilon(\mathbf{r})\,\frac{\omega^2}{c^2}\,E + \nabla^2 E = \frac{\omega^2}{c^2}\,\chi^{(3)}|E|^2 E$$

inhomogeneous dielectric structure

Kerr nonlinearity



Lahini *et al.*, *PRL* **100**, 013906 (2008)

inhomogeneous dielectric structure → transverse eigenmodes

$$E(\mathbf{r}) = \sum_n \psi_n(z)\,u_\perp(x - x_n, y)\,e^{ikz}$$

$k \gg \partial/\partial x, \partial/\partial y$ paraxial approximation

discrete nonlinear Schrödinger equation

$$2ik\,\frac{\partial \psi_n}{\partial z} = q_{\perp,n}^2 \psi_n + \dots$$

Anderson localization and nonlinearity

$$i\frac{d\psi_n}{dt} = \omega_n \psi_n - \Omega(\psi_{n+1} + \psi_{n-1}) + g\psi_n^* \psi_n^2$$

disorder coupling nonlinearity

Anderson localization

$$i\frac{d\psi_n}{dt} = \underbrace{\omega_n\psi_n}_{\text{disorder}} - \underbrace{\Omega(\psi_{n+1} + \psi_{n-1})}_{\text{coupling}}$$

Anderson localization

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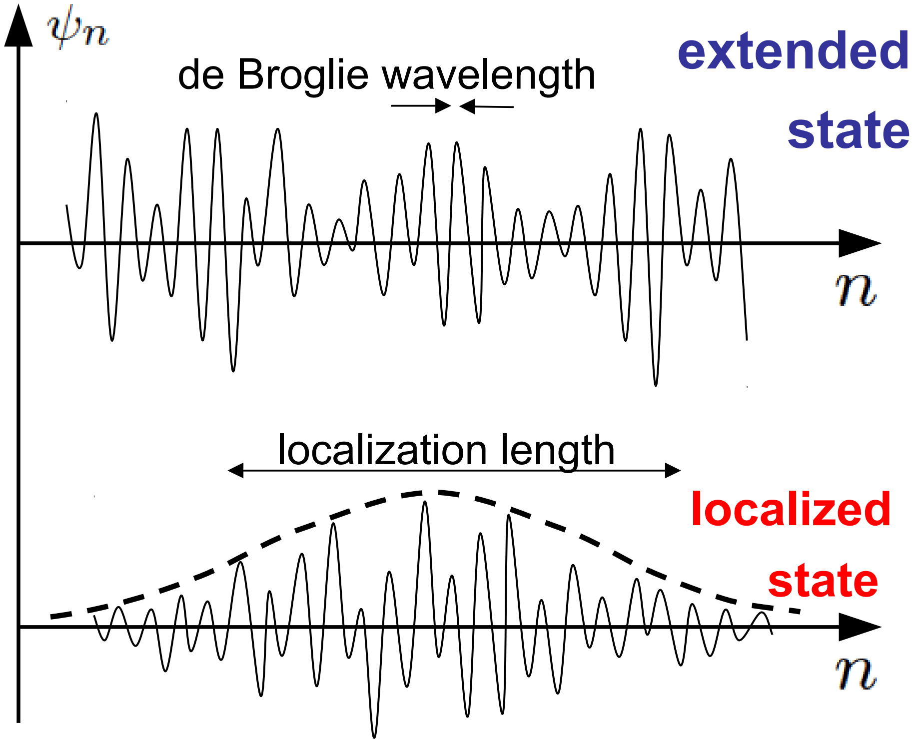
In one dimension
all eigenstates
are localized



Weak disorder: plane waves

↓

mean free path

↓

localization length (= 2 × m.f.p.)

Strong disorder:

localization on one site

↓

weak perturbative tails
on the neighboring sites

Anderson localization and nonlinearity

$$i\frac{d\psi_n}{dt} = \omega_n\psi_n - \Omega(\psi_{n+1} + \psi_{n-1}) + g\psi_n^*\psi_n^2$$

disorder coupling nonlinearity

STRONG disorder + WEAK coupling
↓
STRONG localization

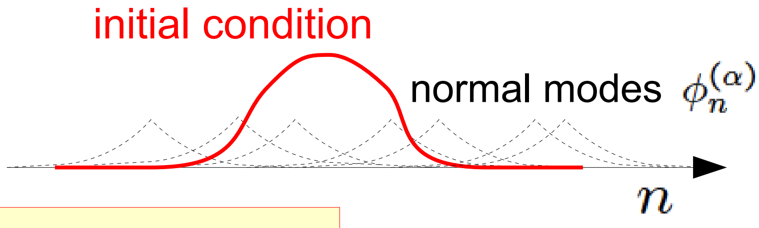
WEAK
nonlinearity

What is the effect of a WEAK nonlinearity
on STRONG Anderson localization?

The problem of wave packet spreading

Linear system: Anderson localization

$$\psi_n(t) = \sum_\alpha c_\alpha \phi_n^{(\alpha)} e^{-i\omega_\alpha t}, \qquad c_\alpha = \sum_n \phi_n^{(\alpha)} \psi_n(0)$$



The wave packet remains exponentially localized forever

Nonlinear system: interaction between the normal modes

$$i\frac{dc_\alpha}{dt} = \sum_{\beta\gamma\delta} V_{\alpha\beta\gamma\delta}\, e^{i(\omega_\alpha+\omega_\beta-i\omega_\gamma-i\omega_\delta)t}\, c_\beta^* c_\gamma c_\delta$$

→ small correction?

→ chaotic behavior?

The problem of wave packet spreading

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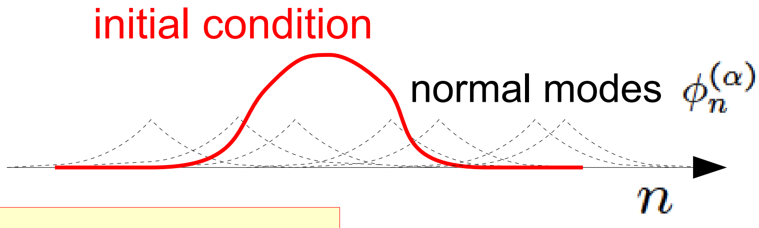

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small correction?

chaotic behavior?

Numerical integration: $\langle \Delta x^2 \rangle \propto t^p$, $p \sim 0.3 - 0.4$ subdiffusion

Shepelyansky (1993); Molina (1998); Kopidakis *et al.* (2008); Pikovsky & Shepelyansky (2008); Skokos *et al.* (2009); Skokos & Flach (2010); Laptyeva *et al.* (2010); Bodyfelt *et al.* (2011)

Indications for slowing down: Mulansky *et al.* (2011); Michaely & Fishman (2012)

KAM theorem, perturbation theory: $p \to 0$

Bourgain & Wang (2008); Wang & Zhang (2009); Fishman *et al.* (2009); Johansson *et al.* (2010)

Experiment: subdiffusion (non-universal exponent) Lucioni *et al.* (2011)

Given

1. Strong localization
2. Weak nonlinearity

$\Big\}$ → worst conditions for transport

3. (Almost) arbitrary initial condition with extensive norm and energy

Question: will the system equilibrate at long distances, and how?

Answer: yes, by normal nonlinear diffusion:

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} D(\rho) \frac{\partial \rho}{\partial x}$$

Mechanism: CHAOS

- concentrated on rare local chaotic spots

  (as seen by Oganesyan, Pal, Huse, 2009)
- which produce stochastic force
- and redistribute the energy between oscillators

Assumptions

$$i\,\frac{d\psi_n}{dt} = \omega_n\psi_n - \Omega(\psi_{n+1} + \psi_{n-1}) + g\psi_n^*\psi_n^2$$

disorder coupling nonlinearity

$$-\frac{W}{2} \le \omega_n \le \frac{W}{2}$$

$$g > 0$$

1. Strong localization: $\frac{\Omega}{W} \equiv \tau \ll 1$

assumption about the Hamiltonian

2. Weak nonlinearity: $\frac{g|\psi_n|^2}{W} \sim \rho \ll 1$

(nonlinear frequency shift << disorder)

note the invariance under $\psi_n \to C\psi_n, \quad g \to C^{-2}g$

3. High temperature: $H < 0, \quad |H| \ll W\sum|\psi_n|^2$

(no superfluid effects)

assumptions about the initial conditions

Thermalization and transport

Two conserved quantities: total energy H, total action $I = \sum |\psi_n|^2$

Local equilibration
in a **finite** time → $\mathcal{P}(\{\psi_n\}) \propto e^{-\beta(H-\mu I)}, \qquad \beta \equiv 1/T$

Global equilibration: transport of the conserved quantities

Macroscopic action density: $\rho(x) = \frac{1}{L^*} \sum_{n=x-L^*/2}^{x+L^*/2} \frac{g|\psi_n|^2}{W} \approx \frac{gT}{|\mu|W}$

length scale to be specified later

get rid of the energy density thanks to $|H| \ll WI$

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} \underset{\textbf{finite!}}{D(\rho)} \frac{\partial \rho}{\partial x}$$

Diffusion coefficient

$$D(\rho) \sim \exp\left(-\mathcal{C}\ln^2\frac{1}{\tau^p\rho}\ln\frac{1}{\rho}\right) \qquad \frac{1}{2} \leq p \leq 3$$

stronger than any
power law

Diffusion coefficient

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stronger than any
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$$\frac{1}{3}\left[1+\ln\left(1+\frac{\ln(1/\rho)}{\ln(1/\tau)}\right)\right]^{-2} \leq \mathcal{C} \leq 8\left[1+\ln\left(1+\frac{\ln(1/\rho)}{\ln(1/\tau)}\right)\right]^{-2}$$

double logarithm ~ constant

D^{-1} is self-averaging at distances $L^* \gg \exp\left(\mathcal{C}\ln^2\frac{1}{\tau^p\rho}\right)$

Basko, *Ann. Phys.* **326**, 1577 (2011)

Probability of chaos

Disorder realization $\{\omega_n\}$

Initial condition $\{\psi_n\}$

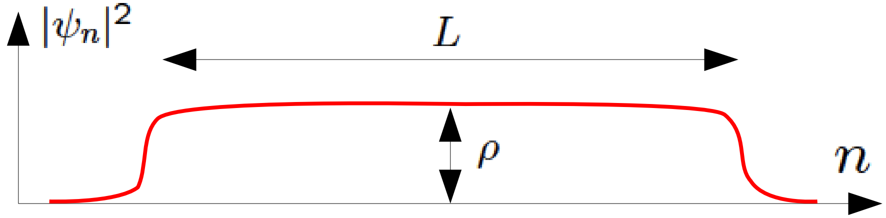
$\rightarrow$ trajectory

regular $\lambda = 0$

chaotic $\lambda > 0$

Lyapunov exponent $\lambda(\psi) = \lim_{t\to\infty} \lim_{\tilde{\psi}(0)\to\psi(0)} \frac{1}{t} \ln \frac{||\tilde{\psi}(t) - \psi(t)||}{||\tilde{\psi}(0) - \psi(0)||}$

Probability to be on a chaotic trajectory:



$$P_L(\tau,\rho) = \int_{-W/2}^{W/2} \prod_{n=1}^{L} \frac{d\omega_n}{W} \int \prod_{n=1}^{L} d^2\psi_n\, \delta\left(|\psi_n|^2 - \rho W/g\right) \Theta_{\mathrm{chaotic}}(\{\omega_n\},\{\psi_n\})$$

or $e^{-g|\psi_n|^2/\rho W}$

1 if chaotic

0 if regular

strong localization

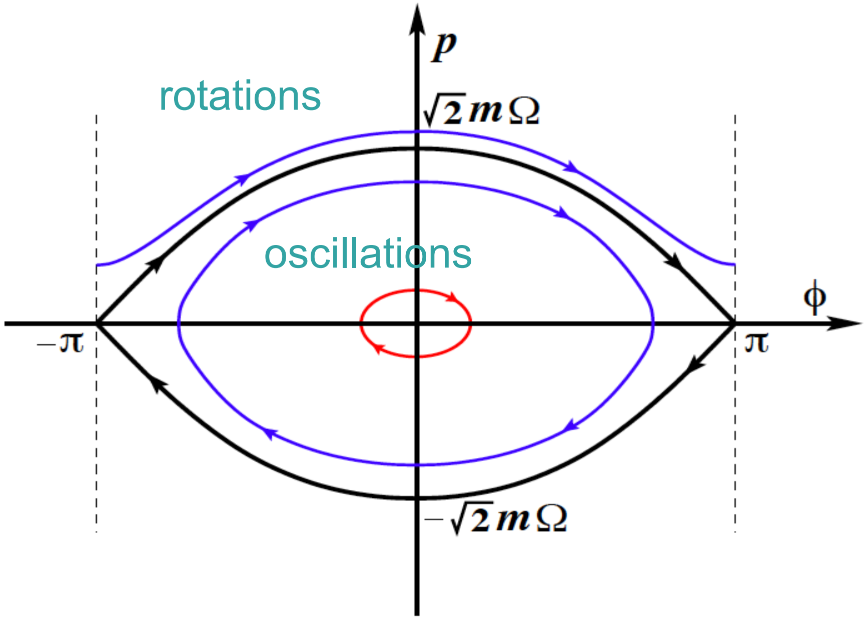
weak nonlinearity

$\tau \ll 1,\ \rho \ll 1$ ⇨ locality in space ⇨ $P_{L\gg 1}(\tau,\rho) = 1 - e^{-w(\tau,\rho)\,L}$

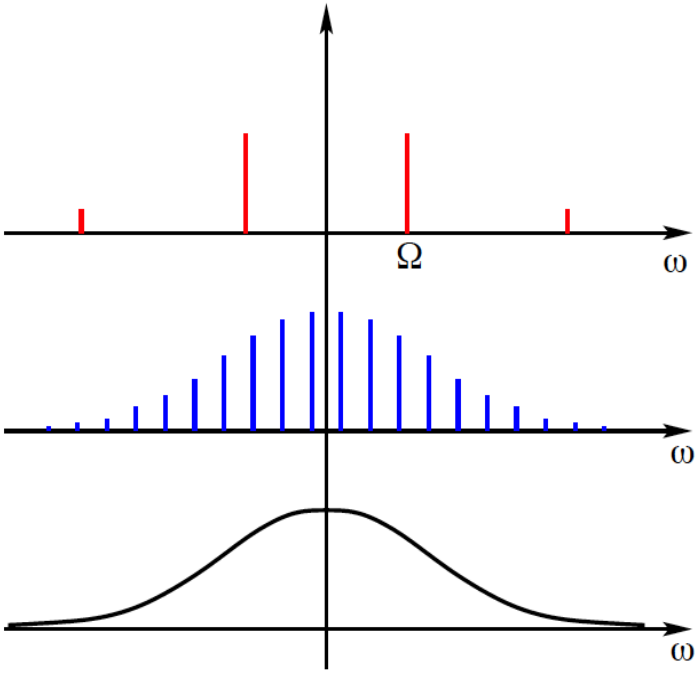
Basko, *Ann. Phys.* **326**, 1577 (2011); *Phys. Rev. E* **86**, 036202 (2012),
Pikovsky & Fishman, *Phys. Rev. E* **83**, 025201 (2011)

Pendulum: $H(p,\phi) = \frac{p^2}{2m} - m\Omega^2 \cos\phi$

Phase space:



Spectrum $\int p(t)\, e^{i\omega t}\, dt$



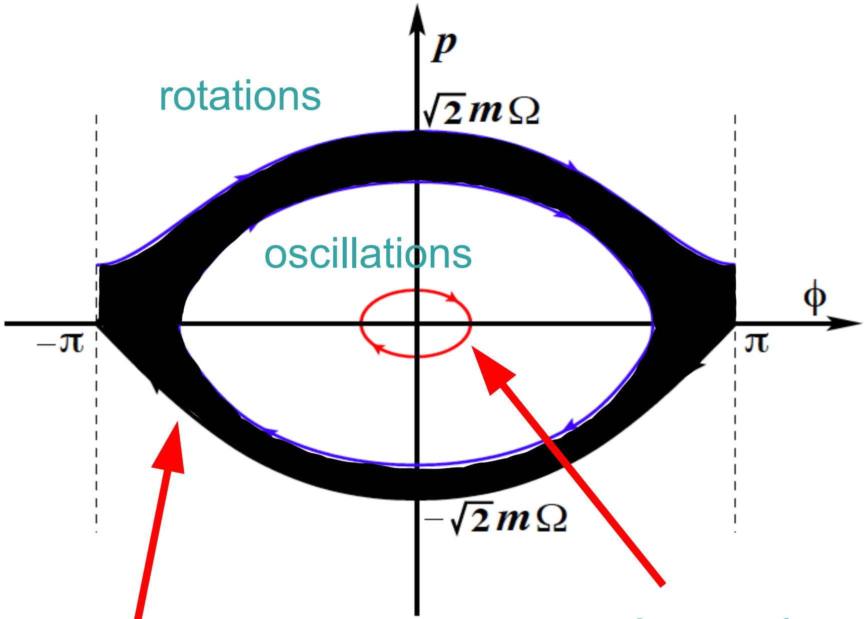
the period diverges
at the separatrix

↔

the separatrix motion
has a continuous spectrum

Perturbed pendulum:

$$H(p,\phi,t)=\frac{p^2}{2m}-m\Omega^2\cos\phi-V\cos(\phi-\omega t)$$



rotations

oscillations

ergodic trajectories
within
the stochastic layer

regular motion
survives

Stochastic layer area:

$$W_s\equiv\int_{\text{layer}}\frac{dp\,d\phi}{2\pi}\sim\frac{V}{\Omega}e^{-|\omega|/\Omega}$$

Melnikov-Arnold integral

$$|\omega|\gg\Omega$$

Continuous spectrum
of the chaotic motion:

$$\left\langle e^{i\phi(t)}e^{-i\phi(t')}\right\rangle_\omega\sim\frac{1}{\Omega}e^{-|\omega|/\Omega}$$

review: B. Chirikov (1979)

Making a pendulum out of oscillators

two-oscillator Hamiltonian:

$$H = \boxed{\omega_1|\psi_1|^2+\frac{g}{2}|\psi_1|^4}+\boxed{\omega_2|\psi_2|^2+\frac{g}{2}|\psi_2|^4}-\boxed{\Omega(\psi_1^*\psi_2+\psi_2^*\psi_1)}$$

action-angle variables: $\psi_n = \sqrt{I_n}\, e^{-i\phi_n}$

$$H = \boxed{\omega_1 I_1 + \frac{gI_1^2}{2}} + \boxed{\omega_2 I_2 + \frac{gI_2^2}{2}} - \boxed{2\Omega\sqrt{I_1 I_2}\cos(\phi_1-\phi_2)}$$

canonical transformation:

$$I = I_1 + I_2, \quad \phi = \frac{\phi_1+\phi_2}{2}, \quad \tilde{I} = \frac{I_1-I_2}{2}, \quad \tilde{\phi} = \phi_1-\phi_2$$

$$H = \underbrace{H_0(I)}_{\text{constant}} + \underbrace{(\omega_1-\omega_2)\tilde{I} + g\tilde{I}^2}_{\text{shift}} - 2\Omega\underbrace{\sqrt{I^2/4-\tilde{I}^2}}_{\text{almost constant}}\cos\tilde{\phi}$$

$\Omega \ll W$

A third oscillator:

$2\Omega\sqrt{I_2 I_3}\cos(\phi_2-\phi_3)$

perturbation of the pendulum

Three oscillators are sufficient to generate chaos

The price of making a pendulum

To find a separatrix:

$$\omega_1 + gI_1 = \omega_2 + gI_2$$

$$gI_1, gI_2 \ll W$$

need $|\omega_1 - \omega_2| \ll W$

Look for a resonance

(guiding resonance)

To create the stochastic layer:

the pendulum frequency $\sim \sqrt{\Omega g I}$

$$\exp\left(-\frac{|\omega_2 - \omega_3|}{\sqrt{\Omega g I}}\right)$$

Look for another resonance

(layer resonance)

Making a pendulum out of more oscillators

$$-\Omega(\psi_1^*\psi_2+\psi_2^*\psi_1)+\omega_2\psi_2^*\psi_2-\Omega(\psi_2^*\psi_3+\psi_3^*\psi_2)$$

effective coupling 1 ↔ 3: $\dfrac{\Omega^2}{\omega_1-\omega_2}(\psi_1^*\psi_3+\psi_3^*\psi_1)$

works when $\omega_1 \approx \omega_3 \neq \omega_2$

Coupling + nonlinearity → effective couplings of the form

$$\psi_1^*\psi_2^*\psi_3^*\psi_4\psi_5\psi_6 \to \cos(\phi_1+\phi_2+\phi_3-\phi_4-\phi_5-\phi_6)$$

Guiding and layer resonances can be generated in high orders of the perturbation theory

Competition: number of combinations ↔ power of the coupling constants

Chaos comes from rare resonant triples: $w \sim \min\left\{\frac{\Omega}{W}\rho, \rho^2\right\}$

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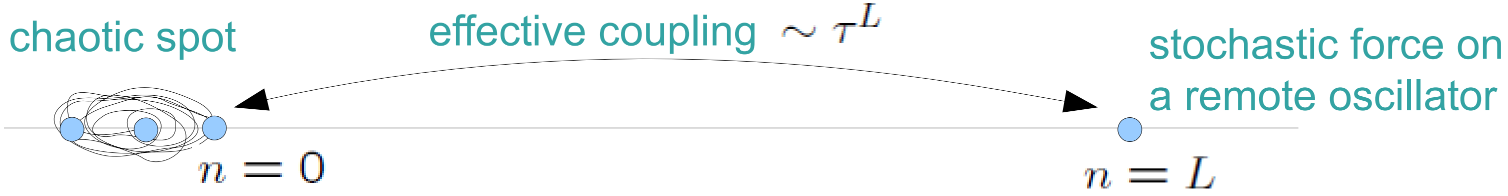
Arnold diffusion



Effective coupling $2V_{m_1 \ldots m_N} \cos(m_1\phi_1 + \ldots + m_N\phi_N)$

action conservation: $m_1 + \ldots + m_N = 0$

Change in actions due to the stochastic force after time t

$$\langle \delta I_n \delta I_{n'} \rangle \sim t\, V^2_{m_1 \ldots m_N} \frac{m_n m_{n'}}{\Omega} \exp\left(-\frac{|m_1\omega_1 + \ldots + m_N\omega_N|}{\Omega} \right)$$

probability = $\underbrace{f(\{I_n\})}_{\text{distribution function}}\, W_s\, g |\vec{m}^{\mathsf{g}}|^2\, \delta\big(\underbrace{m_1^{\mathsf{g}}(\omega_{n_1^{\mathsf{g}}} + gI_{n_1^{\mathsf{g}}}) + \ldots + m_N^{\mathsf{g}}(\omega_{n_N^{\mathsf{g}}} + gI_{n_N^{\mathsf{g}}})}_{\text{guiding resonance}}\big) \prod_n dI_n$

stochastic layer area

$$W_s \frac{\partial f}{\partial t} = \sum_{n,n'} \frac{\partial}{\partial I_n} W_s D_{nn'} \frac{\partial f}{\partial I_{n'}}$$

Constraints on the diffusion:

1. Total action is conserved

2. Total energy is conserved

3. The system stays on the guiding resonance

Conclusions

1. Strong Anderson localization + weak nonlinearity → weak chaos
2. Rare chaotic spots play the role of a bath
3. They produce transport by driving the Arnold diffusion