Berry phase of elastic waves
Experimental study and stochastic model

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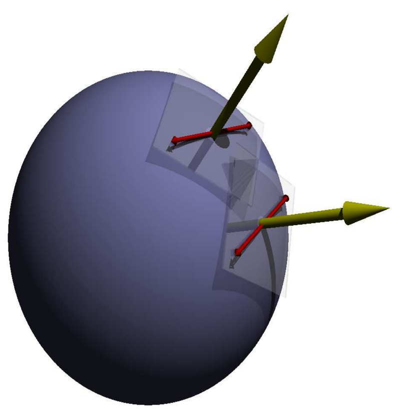
Polarization and Berry phase of elastic waves: observation

• Berry phase, elastic waves and polarization

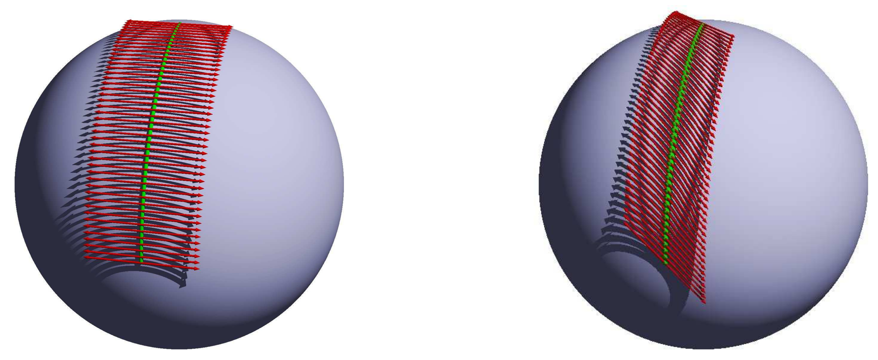
Direction of propagation & polarization



• Direction of propagation
↪ Element of $\mathcal{S}^2$
• Polarization **v**
↪ Vector of $T\mathcal{S}^2$

Polarization vector $\mathbf{v}$ evolves in the tangent bundle of $\mathcal{S}^2$.

Parallel transport

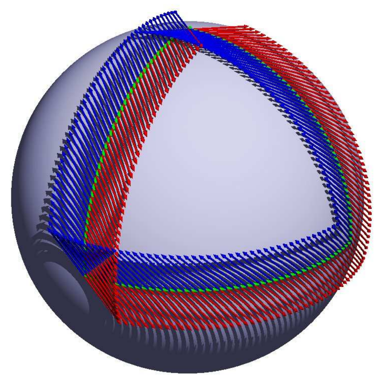


Polarization $\mathbf{v}$ is parallel transported over the unit sphere $\mathcal{S}^2$.

$$\Longrightarrow \frac{D\mathbf{v}}{ds} = \frac{d\mathbf{v}}{ds} - (\frac{d\mathbf{v}}{ds}.\hat{\mathbf{t}})\hat{\mathbf{t}} = 0$$

with s the arc length and $\hat{\mathbf{t}}$ the vector tangent to the path.

Holonomy

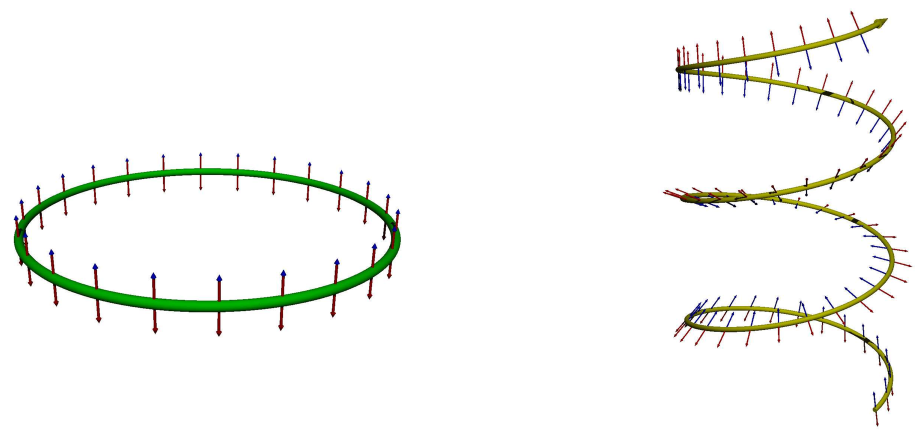


Phase difference Φ after parrallel transport of $\mathbf{v}$ around a close area:

Φ = Area surrounded by the closed path on $\mathcal{S}^2$

Fueller theorem

⋆ Compute the geometric phase between reference trajectory (circle, green path) and the bent trajectory (helix, yellow path).



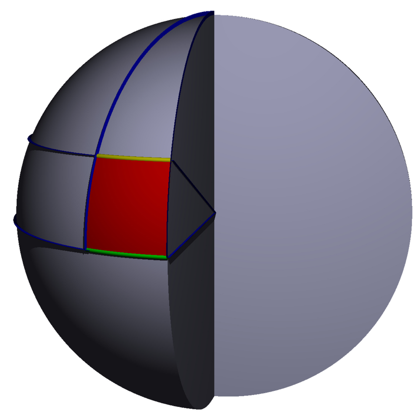
⋆ Geometric phase Φ is:

$$\Phi = \frac{s}{\Lambda}\sin\alpha$$

with α the angle of $\hat{\mathbf{t}}$ with respect to the horizontal direction and Λ the length of a turn of the helix.

Geometrically:

⋆**Green path**: path of the direction of propagation on the circular trajectory
⋆**Yellow path**: path of the direction of propagation on the helix trajectory
⋆**Red area**: geometric phase difference obtained by Fuller's formula.



Frénet-Serret formulas

The intrinsic movement is described by:

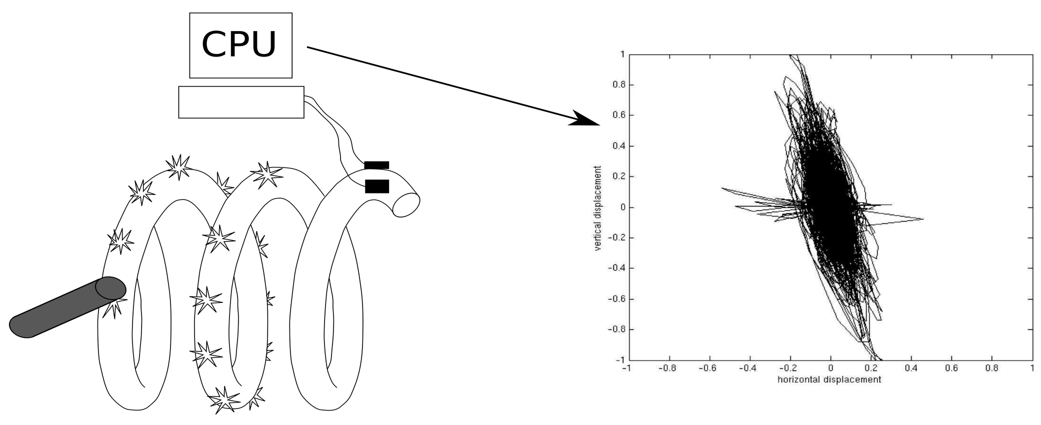
$$\frac{d\hat{\mathbf{n}}}{ds} = -\kappa\hat{\mathbf{t}} + \tau\hat{\mathbf{b}} \quad \text{and} \quad \frac{d\hat{\mathbf{b}}}{ds} = -\tau\hat{\mathbf{n}}$$

where $\tau = \sin\alpha/\Lambda$ is the *torsion* and $\kappa = \cos\theta/\Lambda$ the *curvature*. From $D\mathbf{v}/ds = 0$, one deduces that the rate of rotation of the parallel transported vector is τ ($= \sin\alpha/\Lambda$ for the helix), and thus: $\Phi = \frac{s}{\Lambda}\sin\alpha$

• Experiment

Experimental set-up

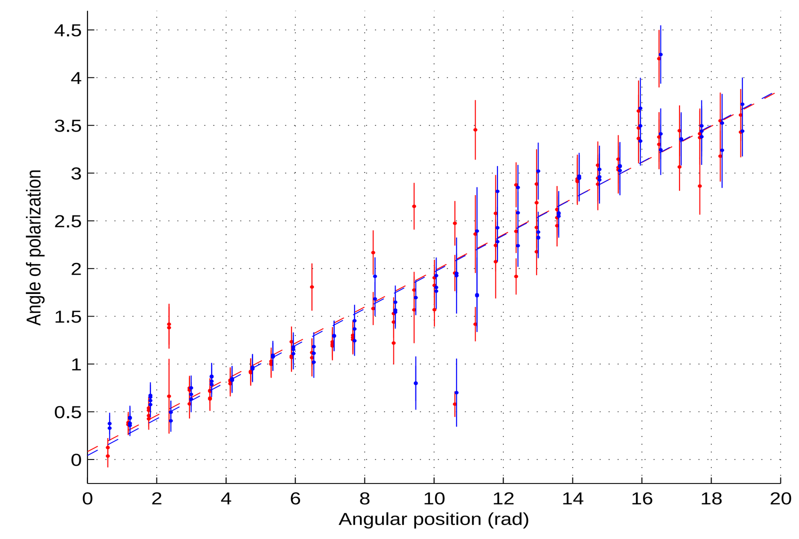
- A steel coil spring as a helix waveguide.
- Generation of flexion waves (linearly polarized) by weak radial hits on the metal bar.
- Two orthogonal accelerometers fixed to the coil spring to record the vibrations of the metal bar.



Parametric plot of directions of vibration + PCA ⇒ Polarization

Results

Measurement of the angle of polarization (*i.e.* the geometric phase) with respect to the angular position of the hitting point along the helix.



Dipsersion of flexion waves ⇒ Evolution of the geometric phase for two different frequencies: 5 kHz (**red**) and 10 kHz (**blue**).

Berry phase distribution of polarization and multiple scattering

• Parallel transport and $SO(3)$

⋆ Parameters for polarized wave study:
- Direction of propagation: $\mathbf{z}$
- Polarization: $\mathbf{v}$

⋆Model:

Polarized wave ⇔ Frame

⋆ Frame:

$$\mathcal{F} = [\mathbf{v}, \mathbf{z}\wedge\mathbf{v}, \mathbf{z}]$$

In the multiple scattering regime, a scattering event modifies:
- The direction of propagation $\mathbf{z}$
- The polarization $\mathbf{v}$ (still orthogonal to $\mathbf{z}$)

Scattering event ⇔ Random rotation $\mathbf{r} = \mathbf{r}_{\psi,\theta,\varphi}$

(Convention: ZYZ for Euler angles)

Parallel transport and rotation

Parallel transport ⇒ 2 degrees of freedom for $\mathbf{r}$

Parallel transport ⇔ $\boxed{\psi = -\varphi}$

• Static process

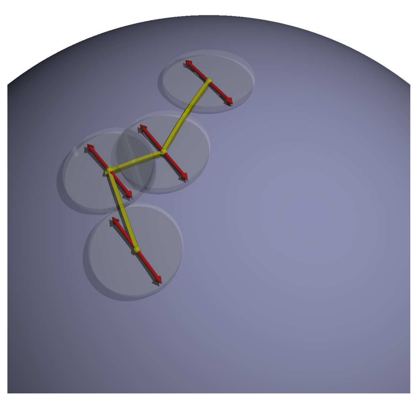
- One scattering event: $\mathcal{F}_1 = \mathcal{P}(\mathcal{F}_0) = \mathcal{F}_0 r_{\psi_0,\theta_0}$
- n successive scattering events: $\mathcal{F}_0 \to \mathcal{F}_n$

$$\mathcal{F}_n = \mathcal{P}^{\bigcirc n}(\mathcal{F}_0) = \mathcal{F}_0\mathbf{r}_{\psi_1,\theta_1}\mathbf{r}_{\psi_2,\theta_2}...\mathbf{r}_{\psi_n,\theta_n} = \mathcal{F}_0\prod_{i=1}^{n}\mathbf{r}_{\psi_i,\theta_i}$$

Multiple scattering + parallel transport
⇕
Right product Levy process on $SO(3)$

Probability density function (pdf) of the frame $\mathcal{F}_n$:

$$p_{\mathcal{F}_n} = p_{\mathcal{F}_0} * p^{*n}_{\mathbf{r}_{\psi,\theta}}$$



Multiple scattering with parallel transport on $SO(3)$

• Dynamical process

Compound Poisson Process

After a time t:

- $n \to N(t)$ (Poisson Process with parameter λ, and λ^{-1}: mean free time)

The process $\mathcal{F}_t$ is:

$$\mathcal{F}_t = \mathcal{F}_0\prod_{i=1}^{N(t)}\mathbf{r}_{\psi_i,\theta_i}$$

Its pdf becomes:

$$p_{\mathcal{F}_t} = \sum_{k=0}^{\infty}\mathbb{P}(N(t)=k)p_{\mathcal{F}_n} = \sum_{k=0}^{\infty}\mathbb{P}(N(t)=k)p^{*n}_{\mathbf{r}_{\psi,\theta}}$$

Initial condition: $p_{\mathcal{F}_0} = \delta(I_3)$

Fourier transform

$p_{\mathcal{F}_t}$ can be decomposed on a Wigner-D basis:

$$\widehat{p}^{l}_{\mathcal{F}_t\,m,n} = < p_{\mathcal{F}_t} \mid D^{l}_{m,n} >$$

where, $\forall l$, $\widehat{p}^{l}_{\mathcal{F}_t\,m,n}$ are elements of $(2l+1)*(2l+1)$ matrices.
- Parallel transport ⇒ $\widehat{p}_{\mathbf{r}_{\psi,\theta}}$ is diagonal
- Multiple convolution + Poisson:

$$\widehat{p}^{l}_{\mathcal{F}_t\,m,n} = e^{\lambda t(\widehat{p}_{\mathbf{r}_{\psi,\theta}}-1)}\delta(m-n)$$

Thanks to the Fourier inverse formulae (with $d^{l}_{m,n}$ the Wigner-d functions):

$$p_{\mathcal{F}_t}(\psi,\theta,\varphi) = \sum_{l}(2l+1)\sum_{|m|\leq l} e^{\lambda(\widehat{p}^{l}_{\mathbf{r}\,m,m}-1)t}d^{l}_{m,m}(\theta)e^{-im(\psi+\varphi)}$$

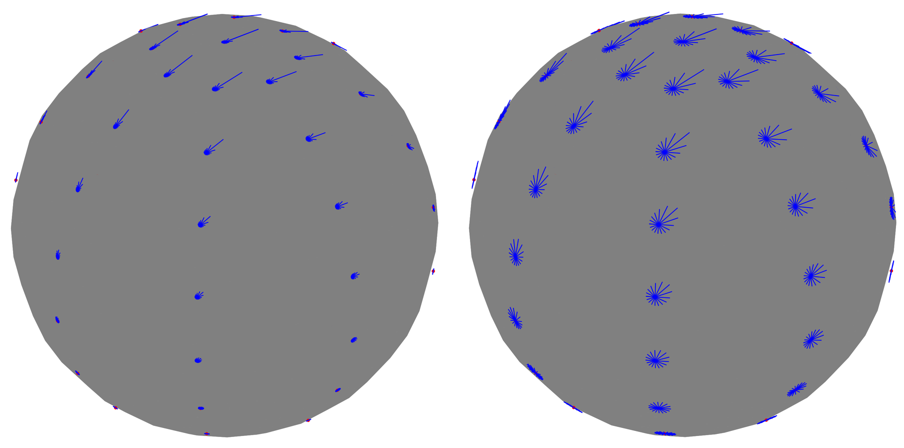
This leads to the distribution of the geometric phase ϕ_g:

$$\boxed{p(\theta,\phi_g) = 2\sum_{m\geq 0}\sum_{l\geq m}(2l+1)e^{\lambda(\widehat{p}^{l}_{\mathbf{r}\,m,m}-1)t}d^{l}_{m,m}(\theta)\cos(m\phi_g)}$$

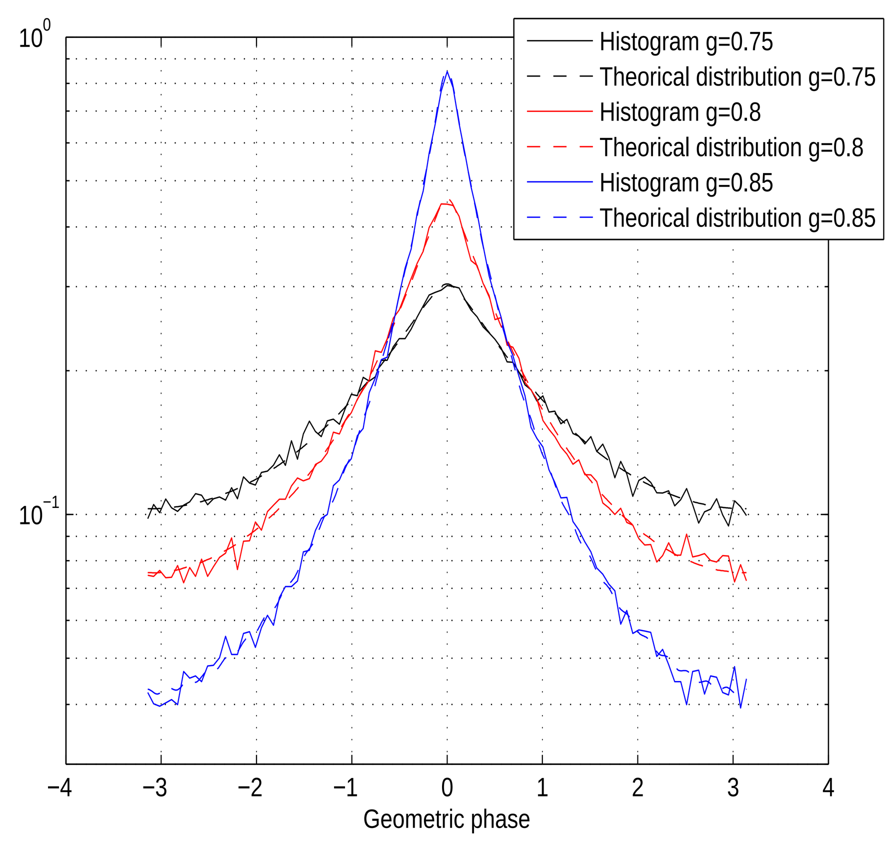
Distribution of the geometric phase
⇕
Scattering phase function

• Simulations

⋆ Distribution of the polarization over $\mathcal{S}^2$ for $\lambda t = 20$ (left) and $\lambda t = 30$ (right)



⋆ Monte Carlo simulation and expression of the distribution of the geometric phase $p(\theta, \phi_g)$ (here given for $\theta < \frac{\pi}{12}$):



• Conclusions

⋆ Observation of the geometric phase for elastic waves
⋆ Stochastic model for geometric phase in multiple scattering regime
⋆ Geometric phase dynamics prediction

• Perspectives

⋆ Observation of the geometric phase in random media
⋆ Statistical inference: parameters estimation for random media