Berry phase of elastic waves
Experimental study and stochastic model

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Polarization and Berry phase of elastic waves: observation

- Berry phase, elastic waves and polarization
  Direction of propagation & polarization

  Polarization vector \( \mathbf{v} \) evolves in the tangent bundle of \( \mathbb{S}^2 \).
  Parallel transport

  Polarization \( \mathbf{v} \) is parallel transported over the unit sphere \( \mathbb{S}^2 \).

  Direction of propagation
  - Element of \( \mathbb{S}^2 \)
  - Polarization \( \mathbf{v} \)
  - Vector of \( T\mathbb{S}^2 \)

  Holonomy

  Phase difference \( \Phi \) after parallel transport of \( \mathbf{v} \) around a close area:

  \[ \Phi = \text{Area surrounded by the closed path on } \mathbb{S}^2 \]

  with \( \alpha \) the angle of \( \mathbf{t} \) with respect to the horizontal direction and \( A \) the length of a turn of the helix.

Geometrically
- Green path: path of the direction of propagation on the circular trajectory
- Yellow path: path of the direction of propagation on the helix trajectory
- Red area: geometric phase difference obtained by Fuller’s formula

Frénet-Serret formulas
The intrinsic movement is described by:

\[ \frac{d\mathbf{a}}{dt} = -\kappa \mathbf{t} + \alpha \mathbf{b} \quad \text{and} \quad \frac{d\mathbf{b}}{dt} = -\kappa \mathbf{t} \]

where \( \tau = \sin\alpha/A \) is the torsion and \( \kappa = \cos\theta/A \) the curvature. From \( D\mathbf{v}/D\mathbf{a} = 0 \), one deduces that the rate of rotation of the parallel transported vector is \( \tau = \sin\alpha/A \) for the helix, and thus

\[ \Phi = \frac{-\kappa}{\sin\alpha} \]

Dynamical process
Compound Poisson Process
After a time \( t \)
- \( n \to X(t) \) (Poisson Process with parameter \( \lambda \) and \( \lambda^{-1} \) mean free time)
The process \( \mathcal{F}_t \)

\[ \mathcal{F}_t = \mathcal{F}_0 \bigoplus \sum_{i=1}^{\infty} \mathcal{F}_{t_i} \]

Its pdf becomes:

\[ p_{\mathbf{F}_t} = \sum_{i=0}^{\infty} p_{X(t)} \]

Initial condition: \( p_{\mathbf{F}_0} = \delta(\mathbf{0}) \)

Fourier transform
\( \mathbf{F}_t \) can be decomposed on a Wigner-D basis:

\[ \tilde{\mathbf{F}}_{t,m,n} = \langle \mathbf{F}_t | \mathcal{D}_{m,n} \rangle \]

where, \( \forall \mathcal{D}_{m,n} \) are elements of \((2l+1) \times (2l+1)\) matrix.
- Parallel transport \( \Rightarrow \mathcal{F}_{t,m,n} \)
- Multiple convolution \( + \) Poisson:

\[ \tilde{\mathbf{F}}_{t,m,n} = \mathcal{F}_{t,m,n} \mathcal{D}_{m,n} \]

Thanks to the Fourier inverse formulae (with \( \hat{d}_{m,n} \) the Wigner-d function):

\[ p_{\mathbf{F}_t}(\mathbf{0},\mathbf{\theta}) = \sum_{m,n=0}^{2l+1} \langle \mathcal{D}_{m,n} | \mathcal{F}_{t,m,n} \rangle \hat{d}_{m,n} \]

This leads to the distribution of the geometric phase \( \Phi_\mathbf{F} \)

\[ \langle \Phi_\mathbf{F} \rangle = 2 \sum_{m,n=0}^{2l+1} \langle \mathcal{D}_{m,n} | \mathcal{F}_{t,m,n} \rangle \hat{d}_{m,n} \]

Distribution of the geometric phase

- Experiment
  - A steel coil spring as a helix waveguide
  - Generation of flexion waves (linearly polarized) by weak radial hits on the metal bar
  - Two orthogonal accelerometers fixed to the coil spring to record the vibrations of the metal bar.

Parametric plot of directions of vibration + PCA \( \Rightarrow \) Polarization

Results
Measurement of the angle of polarization (i.e. the geometric phase) with respect to the angular position of the hitting point along the helix.

Evolution of the geometric phase for two different frequencies: 5 kHz (red) and 10 kHz (blue).

Perspectives
- Observation of the geometric phase in random media
- Statistical inference: parameters estimation for random media