Effect of nonlinearity on Anderson localization of classical waves

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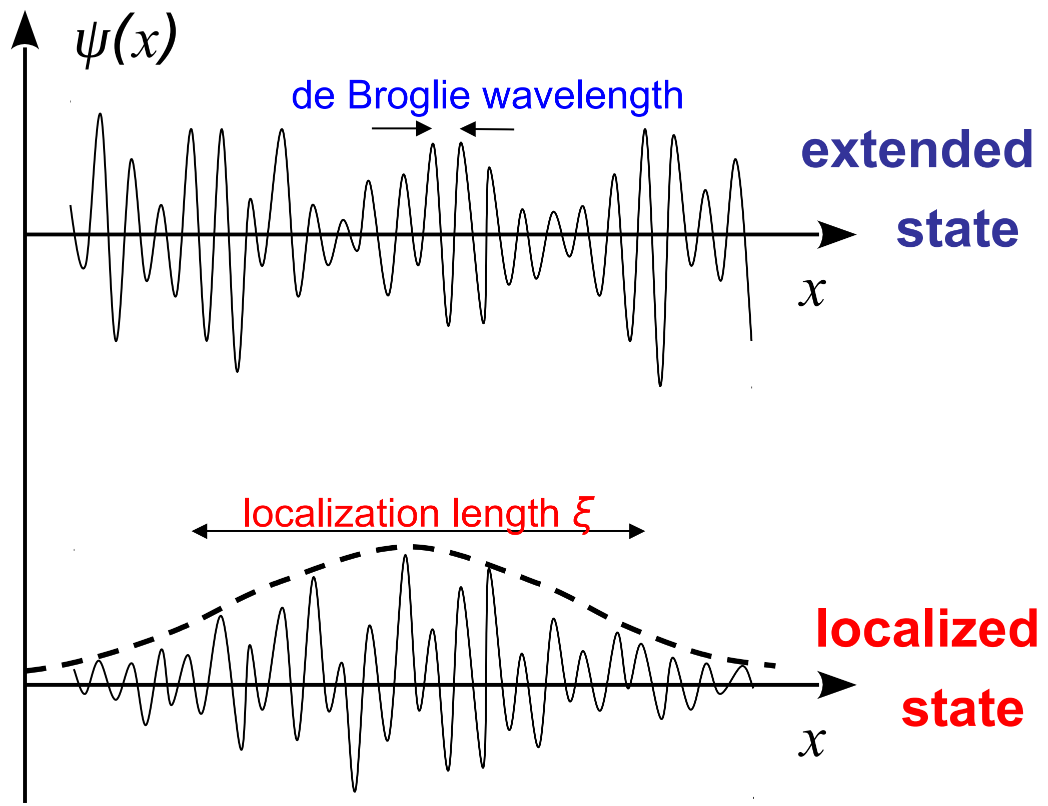
PHYSICAL REVIEW VOLUME 109, NUMBER 5 MARCH 1, 1958

Absence of Diffusion in Certain Random Lattices

P. W. ANDERSON
Bell Telephone Laboratories, Murray Hill, New Jersey



Outline

Lecture 1

- Nonlinear dynamical systems, integrability and chaos
- Statistics of chaos in disordered nonlinear chains

Lecture 2

- Statistical physics of chaotic systems
- Transport in disordered nonlinear chains

Disordered nonlinear wave equations

disorder

+

nonlinearity

continuous

$$\frac{\varepsilon(x)}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{\partial^2 E}{\partial x^2} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\chi^{(3)} E^3\right)$$

$$i\hbar\frac{\partial \Psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\Psi + g|\Psi|^2\Psi$$

discrete lattice

$$i\frac{d\psi_n}{dt} = \epsilon_n\psi_n - \Omega\left(\psi_{n-1} + \psi_{n+1}\right) + g|\psi_n|^2\psi_n$$

$$\frac{d^2u_n}{dt^2} = -\epsilon_n^2 u_n + \Omega^2\left(u_{n+1} - 2u_n + u_{n-1}\right) - g u_n^3$$

In one spatial dimension
all linear normal modes are localized

Superposition principle in linear systems

Lattice with L sites: $n = -L/2+1, \ldots, L/2$, $\quad \vec{\psi} = (\psi_{-L/2+1}, \ldots, \psi_{L/2})$

$$i\,\frac{d\vec{\psi}}{dt} = \hat{\mathcal{L}}\vec{\psi}$$

linear Hermitian operator

Algorithm:

1. Find the eigenfunctions and eigenvalues

$$\hat{\mathcal{L}}\vec{\phi}_\alpha = \omega_\alpha \vec{\phi}_\alpha$$

2. For any initial condition, the solution is

$$\vec{\psi}(t) = \sum_{\alpha=1}^{L} c_\alpha \vec{\phi}_\alpha e^{-i\omega_\alpha t}, \qquad c_\alpha = \sum_n \phi^*_{\alpha n}\psi_n(0)$$

Linear systems: finding L independent periodic solutions $\vec{\psi}(t) = \vec{\phi}\, e^{-i\omega t}$ gives access to all possible solutions

Nonlinear systems: - localized periodic solutions still exist

Fröhlich, Spencer & Wayne, *J. Stat. Phys.* 42, 247 (1986),

- their localization length is the same as in the linear case

Iomin & Fishman, *Phys. Rev. E* **76**, 056607 (2007),

but they are a just small minority among all possible solutions!

Lyapunov exponent

Choose an initial condition $\vec{\psi}(0)$ → trajectory $\vec{\psi}(t)$

A nearby initial condition $\vec{\psi}^{\varepsilon}(0) = \vec{\psi}(0) + \vec{d}$ → trajectory $\vec{\psi}^{\varepsilon}(t)$

Lyapunov exponent

$$\lambda = \lim_{t\to\infty} \lim_{\varepsilon\to 0} \frac{1}{t} \ln \frac{|\vec{\psi}^{\varepsilon}(t) - \vec{\psi}(t)|}{\varepsilon}$$

mind the order of limits

$$|\vec{\psi}^{\varepsilon}(t) - \vec{\psi}(t)| \sim \varepsilon\, e^{\lambda t}$$

initial deviation grows exponentially

Numerical simulation with double precision: $\varepsilon \sim 10^{-16}$

reliable for times $t < 36/\lambda$

λ = 0 for linear systems:

$$\left| \sum_{\alpha} d_{\alpha} \vec{\phi}_{\alpha} e^{-i\omega_{\alpha} t} \right| \leqslant \sum_{\alpha} |d_{\alpha}| |\vec{\phi}_{\alpha}|$$

no exponential growth

Also, λ = 0 for (quasi)periodic solutions of nonlinear equations

Hamiltonian systems

Phase space: coordinates $\mathbf{q} = (q_1,\ldots,q_N)$, momenta $\mathbf{p} = (p_1,\ldots,p_N)$

Hamiltonian function $H(\mathbf{q},\mathbf{p})$ → equations of motion:

$$\frac{dq_n}{dt} = \frac{\partial H(\mathbf{q},\mathbf{p})}{\partial p_n}, \qquad \frac{dp_n}{dt} = -\frac{\partial H(\mathbf{q},\mathbf{p})}{\partial q_n}$$

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Example 1: DNLS chain with disorder

$$H(\boldsymbol{\psi}, i\boldsymbol{\psi}^*) = \sum_n \left[\epsilon_n \psi_n^* \psi_n - \Omega\left(\psi_n^* \psi_{n+1} + \psi_{n+1}^* \psi_n\right) + \frac{g}{2}\, \psi_n^* \psi_n^* \psi_n \psi_n\right]$$

$$\frac{d\psi_n}{dt} = \frac{\partial H}{\partial (i\psi_n^*)} \quad \Rightarrow \quad i\,\frac{d\psi_n}{dt} = \epsilon_n \psi_n - \Omega\left(\psi_{n-1} + \psi_{n+1}\right) + g|\psi_n|^2 \psi_n$$

$$\frac{d(i\psi_n^*)}{dt} = -\frac{\partial H}{\partial \psi_n} \quad \Rightarrow$$ its complex conjugate

Hamiltonian systems

Phase space: coordinates $\mathbf{q} = (q_1,\ldots,q_N)$, momenta $\mathbf{p} = (p_1,\ldots,p_N)$

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$$\frac{dq_n}{dt} = \frac{\partial H(\mathbf{q},\mathbf{p})}{\partial p_n}, \qquad \frac{dp_n}{dt} = -\frac{\partial H(\mathbf{q},\mathbf{p})}{\partial q_n}$$

Example 2: continuous NLS equation with disorder

$$H[\mathbf{\Psi}, i\hbar\mathbf{\Psi}^*] = \int dx \left(\frac{\hbar^2}{2m}\frac{\partial\Psi^*}{\partial x}\frac{\partial\Psi}{\partial x} + V\Psi^*\Psi + \frac{g}{2}\Psi^*\Psi^*\Psi\Psi \right)$$

$$\frac{\partial\Psi(x)}{\partial t} = \frac{\delta H}{\delta(i\hbar\Psi^*(x))} \quad \Rightarrow \quad i\hbar\frac{\partial\Psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V\right]\Psi + g|\Psi|^2\Psi$$

$$\frac{\partial(i\hbar\Psi^*(x))}{\partial t} = -\frac{\delta H}{\delta\Psi(x)} \quad \Rightarrow$$

its complex conjugate

Integrable Hamiltonian systems

Integral of motion $\mathcal{I}(\mathbf{p},\mathbf{q})\ :\ \frac{d\mathcal{I}}{dt} = \{\mathcal{I}, H\} = 0$

Poisson brackets $\{f,g\} = \sum_n \left(\frac{\partial f}{\partial q_n}\frac{\partial g}{\partial p_n} - \frac{\partial f}{\partial p_n}\frac{\partial g}{\partial q_n} \right)$

Completely integrable system: N degrees of freedom, N integrals of motion

$$\mathcal{I}_1, \ldots, \mathcal{I}_N\ :\ \{\mathcal{I}_\alpha, \mathcal{I}_\beta\} = 0$$

Canonical transformation $(\mathbf{q},\mathbf{p}) \to (\vec{\mathcal{I}}, \vec{\vartheta})$: choose $\mathcal{I}_1, \ldots, \mathcal{I}_N$ as new momenta

$$\frac{d\mathcal{I}_\alpha}{dt} = -\frac{\partial H}{\partial \vartheta_\alpha} = 0$$
$$\frac{d\vartheta_\alpha}{dt} = \frac{\partial H}{\partial \mathcal{I}_\alpha} \equiv \omega_\alpha(\vec{\mathcal{I}})$$

$$\mathcal{I}_\alpha = \mathrm{const}$$
$$\vartheta_\alpha(t) = \vartheta_\alpha^0 + \omega_\alpha t$$

Lyapunov exponent = 0

Invariant tori

The choice of $\mathcal{I}_1, \ldots, \mathcal{I}_N$ is not unique

Confined motion: choose $\vec{\mathcal{I}}$ so that for all α

$$\mathbf{q}(\vartheta_\alpha) = \mathbf{q}(\vartheta_\alpha + 2\pi), \quad \mathbf{p}(\vartheta_\alpha) = \mathbf{p}(\vartheta_\alpha + 2\pi)$$

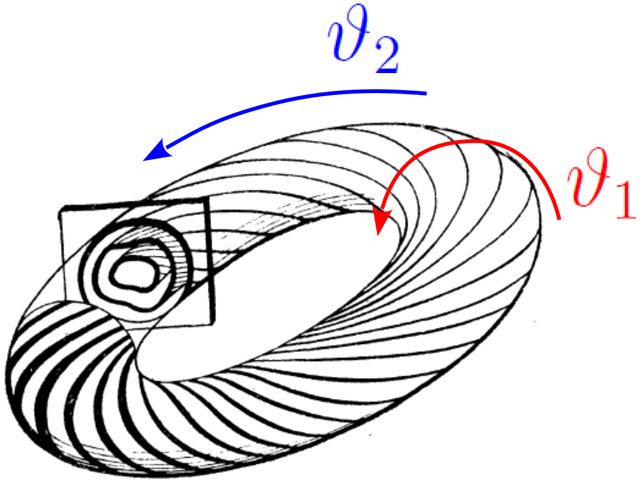
$(\vec{\mathcal{I}}, \vec{\vartheta})$ – action-angle variables

$$\mathcal{I}_\alpha = \text{const}$$

$$\vartheta_\alpha(t) = \vartheta_\alpha^0 + \omega_\alpha t$$

motion on an N-dimensional torus

in the $2N$-dimensional phase space



From V. I. Arnold, *Usp. Mat. Nauk* **18**, 13 (1963)

Incommensurate $\omega_1, \ldots, \omega_N$ ⇒ Trajectory covers the whole torus

Examples of integrable systems

1. Any linear system: projections on eigenvectors are conserved
2. Any conservative system with 1 degree of freedom: energy is conserved

3. Two-site DNLS chain:

$$H = \epsilon_1 \psi_1^* \psi_1 + \frac{g}{2} \psi_1^* \psi_1^* \psi_1 \psi_1 + \epsilon_2 \psi_2^* \psi_2 + \frac{g}{2} \psi_2^* \psi_2^* \psi_2 \psi_2 - \Omega \left(\psi_1^* \psi_2 + \psi_2^* \psi_1 \right)$$

two conserved quantities: energy and norm $\mathcal{N} = |\psi_1|^2 + |\psi_2|^2$

4. Clean continuous NLS eq.: $H = \int dx \left(\frac{\hbar^2}{2m} \frac{\partial \Psi^*}{\partial x} \frac{\partial \Psi}{\partial x} + \frac{g}{2} \Psi^* \Psi^* \Psi \Psi \right)$

$$C_1 = \int |\Psi|^2 \, dx, \quad C_2 = \int \left(\Psi^* \partial_x \Psi - \Psi \partial_x \Psi^* \right) dx, \quad C_3 = H,$$

$$C_4 = \int \left(\Psi \partial_x^3 \Psi^* - \frac{3g}{2} |\Psi|^2 \Psi \partial_x \Psi^* \right) dx,$$

$$C_5 = \int \left[|\partial_x^2 \Psi|^2 + \frac{g^2}{2} |\Psi|^6 + \frac{g}{2} \left(\partial_x |\Psi|^2 \right)^2 + 3g |\Psi|^2 |\partial_x \Psi|^2 \right] dx,$$

$$\ldots$$

complete infinite set of conserved quantities

Zakharov & Shabat, *JETP* **34**, 62 (1972)

Examples of integrable systems

5. Toda lattice: $H = \sum_n \left(\frac{p_n^2}{2} + e^{q_n - q_{n+1}} \right)$ M. Toda, *J. Phys. Soc. Jpn.* **22**, 431 (1967)

6. Ablowitz-Ladik chain:

$$H = \sum_n \left[-\Omega \left(\psi_n^* \psi_{n+1} + \psi_{n+1}^* \psi_n \right) + \frac{g}{2} |\psi_n|^2 |\psi_{n+1}|^2 \right]$$

Ablowitz & Ladik, *J. Math. Phys*. **16**, 698 (1975)

Clean DNLS chain: **non-integrable!**

$$H = \sum_n \left[-\Omega \left(\psi_n^* \psi_{n+1} + \psi_{n+1}^* \psi_n \right) + \frac{g}{2} |\psi_n|^4 \right]$$

Kolmogorov-Arnold-Moser theorem

$$H(\vec{\mathcal{I}}, \vec{\vartheta}\,) = H_0(\vec{I}\,) + \varepsilon\, V(\vec{\mathcal{I}}, \vec{\vartheta}\,)$$

integrable

integrability-breaking perturbation

small parameter

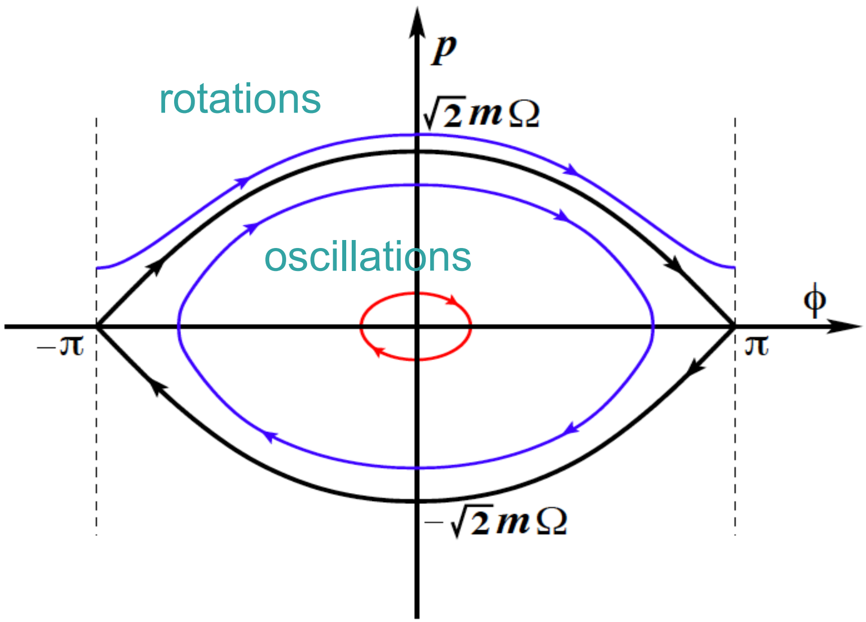
$$\det \left| \frac{\partial^2 H_0}{\partial \mathcal{I}_\alpha \partial \mathcal{I}_\beta} \right| \neq 0$$

At small ε most of the tori are preserved,
measure of destroyed tori → 0 as ε → 0

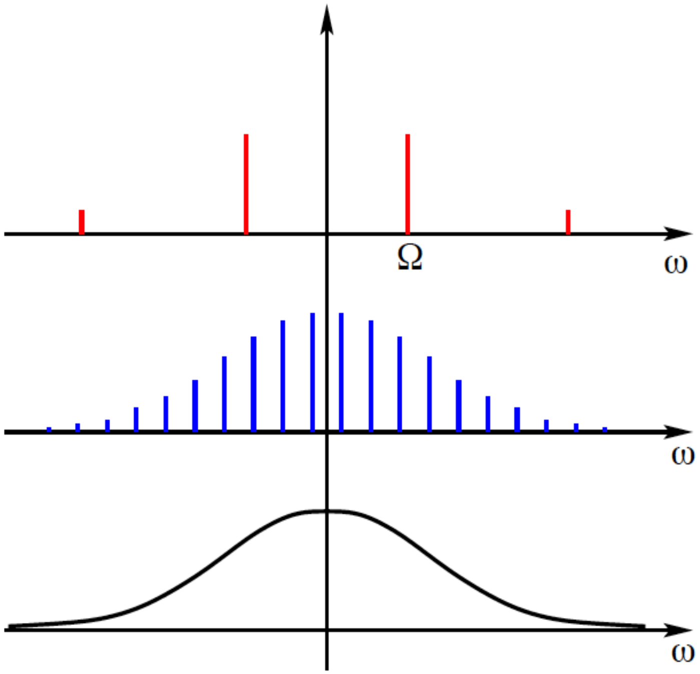
The phase space splits into regular and chaotic regions

Pendulum: $H(p,\phi) = \frac{p^2}{2m} - m\Omega^2 \cos\phi$

Phase space:



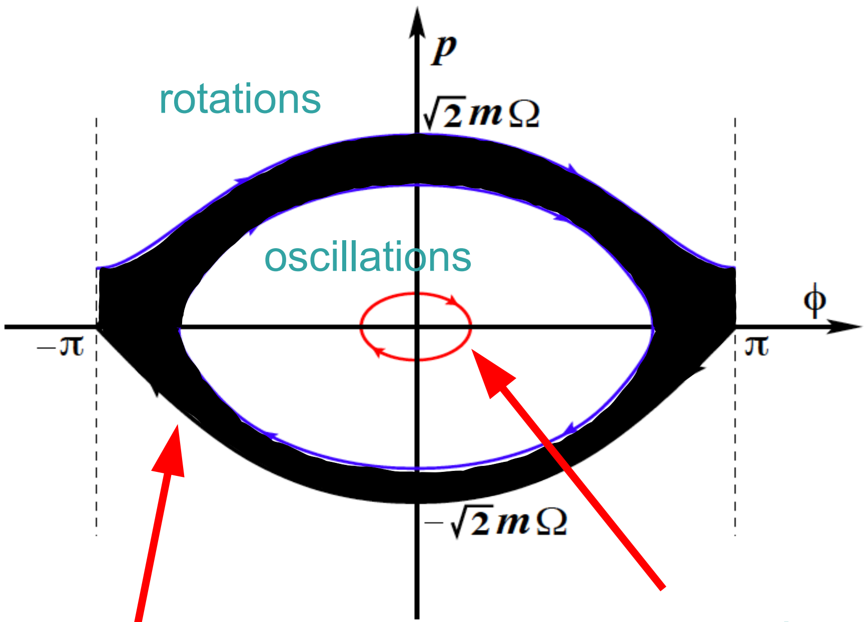
Spectrum $\int p(t)\, e^{i\omega t}\, dt$



the period diverges at the separatrix ⟺ the separatrix motion has a continuous spectrum

Perturbed pendulum:

$$H(p,\phi,t) = \frac{p^2}{2m} - m\Omega^2 \cos\phi - V\cos(\phi - \omega t)$$



ergodic trajectories within the stochastic layer

regular motion survives

Stochastic layer area:

$$W_s \equiv \int_{\text{layer}} \frac{dp\,d\phi}{2\pi} \sim \frac{V}{\Omega} e^{-|\omega|/\Omega}$$

Melnikov-Arnold integral

$$|\omega| \gg \Omega$$

Continuous spectrum of the chaotic motion:

$$\left\langle e^{i\phi(t)} e^{-i\phi(t')} \right\rangle_\omega \sim \frac{1}{\Omega} e^{-|\omega|/\Omega}$$

review: B. Chirikov (1979)

When and how does chaos appear in disordered nonlinear chains?

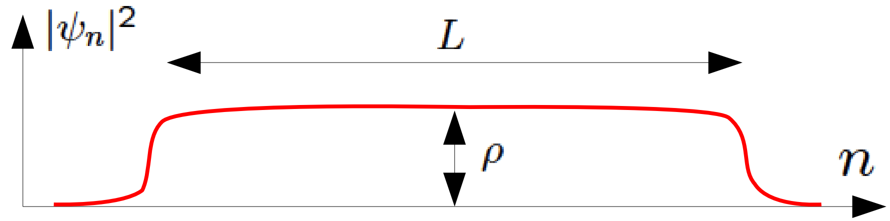
Probability of chaos

$$i\,\frac{d\psi_n}{dt} = \epsilon_n \psi_n - \Omega\left(\psi_{n-1} + \psi_{n+1}\right) + g|\psi_n|^2\psi_n$$

Disorder realization $\{\epsilon_n\}$
Initial condition $\{\psi_n\}$
→ trajectory
- regular $\lambda = 0$
- chaotic $\lambda > 0$

Lyapunov exponent

Initial condition over a length L:



Probability to be on a chaotic trajectory for a chain of length L:

$$P_L(\rho) = \int_{-W/2}^{W/2} \prod_{n=1}^{L} \frac{d\epsilon_n}{W} \int \prod_{n=1}^{L} d^2\psi_n\, \delta\left(|\psi_n|^2 - \rho\right) \Theta_{\text{chaotic}}(\{\epsilon_n\}, \{\psi_n\})$$

or $e^{-|\psi_n|^2/\rho}$

0 for regular, 1 for chaotic

disorder average

box distribution

Locality of chaos

$$i\,\frac{d\psi_n}{dt} = \epsilon_n \psi_n - \Omega\,(\psi_{n-1} + \psi_{n+1}) + g|\psi_n|^2 \psi_n$$

Switch to localized normal modes: $\psi_n(t) = \sum_\alpha c_\alpha(t)\,\phi_{\alpha n}\,e^{-i\omega_\alpha t}$

Nonlinearity couples the modes: $i\,\frac{dc_\alpha}{dt} = \sum_{\beta\gamma\delta} e^{i(\omega_\alpha+\omega_\beta-\omega_\gamma-\omega_\delta)t} V_{\alpha\beta\gamma\delta} c_\beta^* c_\gamma c_\delta$

$$V_{\alpha\beta\gamma\delta} = g \sum_n \phi_{\alpha n}\phi_{\beta n}\phi_{\gamma n}\phi_{\delta n}$$

overlap between the modes **only not too far apart**

Transition to chaos should occur independently in different spatial regions

strong localization, weak nonlinearity

$\Omega \ll W\,,\ g\rho \ll W$ ⇒ locality in space ⇒

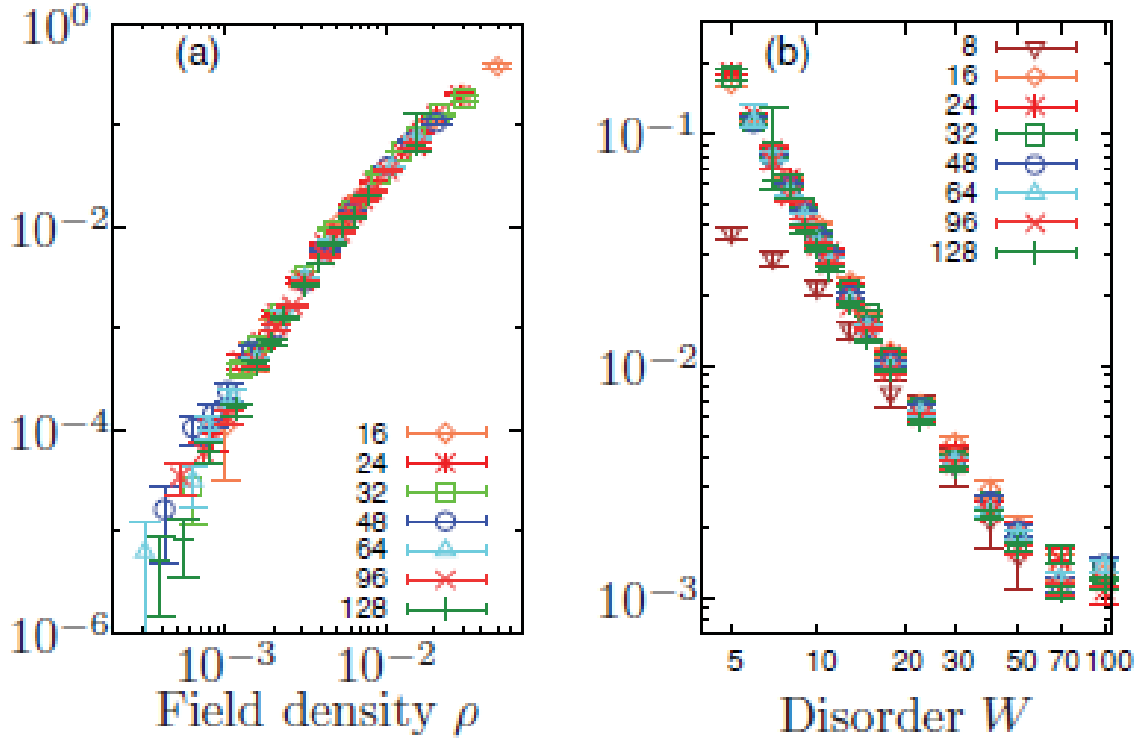
Hypothesis:

$1 - P_{L\gg 1}(\rho) = e^{-w(\rho)\,L}$

Basko, *Ann. Phys.* **326**, 1577 (2011), Pikovsky & Fishman, *Phys. Rev. E* **83**, 025201 (2011)

Direct numerics

$$\frac{1}{L} \ln \frac{1}{1 - P_L}$$



$$w \propto \rho^{2.25} \left(\frac{\Omega}{W}\right)^{2.19}$$

Pikovsky & Fishman, *Phys. Rev. E* **83**, 025201 (2011)

Pendulum in the DNLS equation

two-site Hamiltonian: $$H = \epsilon_1|\psi_1|^2 + \frac{g}{2}|\psi_1|^4 + \epsilon_2|\psi_2|^2 + \frac{g}{2}|\psi_2|^4 - \Omega\left(\psi_1^*\psi_2 + \psi_2^*\psi_1\right)$$

canonical transformation: $$\psi_n = \sqrt{I_n}e^{-i\theta_n}$$

transformed Hamiltonian: $$H = \epsilon_1 I_1 + \frac{g}{2}I_1^2 + \epsilon_2 I_2 + \frac{g}{2}I_2^2 - 2\Omega\sqrt{I_1 I_2}\cos(\theta_1 - \theta_2)$$

another canonical transformation: $$I = I_1 + I_2, \quad \theta = \frac{\theta_1 + \theta_2}{2}, \quad \tilde{I} = \frac{I_1 - I_2}{2}, \quad \tilde{\theta} = \theta_1 - \theta_2$$

(I: conserved)

$$H = H_0(I) + (\epsilon_1 - \epsilon_2)\tilde{I} + g\tilde{I}^2 - 2\Omega\sqrt{I^2/4 - \tilde{I}^2}\cos\tilde{\theta}$$

very similar to a pendulum

$H_0(I)$: constant

$(\epsilon_1 - \epsilon_2)\tilde{I} + g\tilde{I}^2$: complete the square

$$\tilde{I} = \frac{\epsilon_2 - \epsilon_1}{2g} + p$$

$I^2/4 - \tilde{I}^2$: almost constant $\Omega \ll W$

A third site:

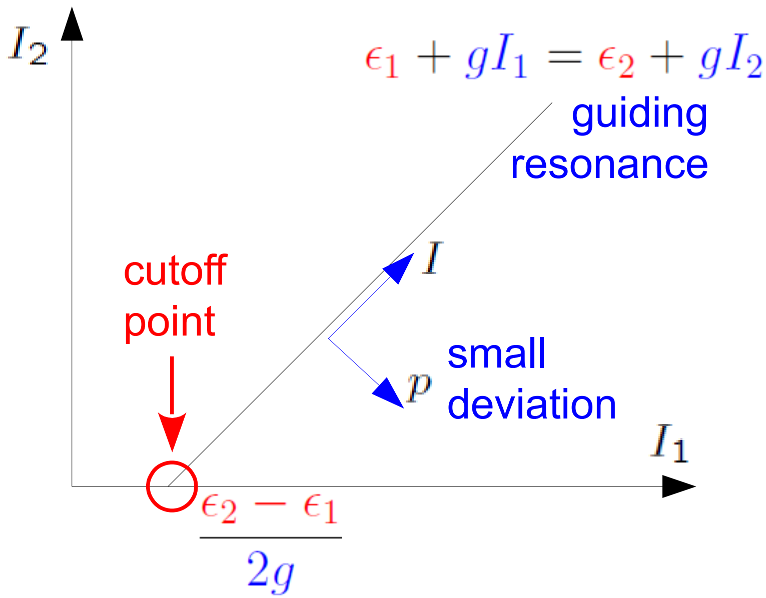
$$V = -2\Omega\sqrt{I_2 I_3}\cos(\theta_2 - \theta_3)$$

perturbation of the pendulum

→ **Three sites are sufficient to generate chaos**

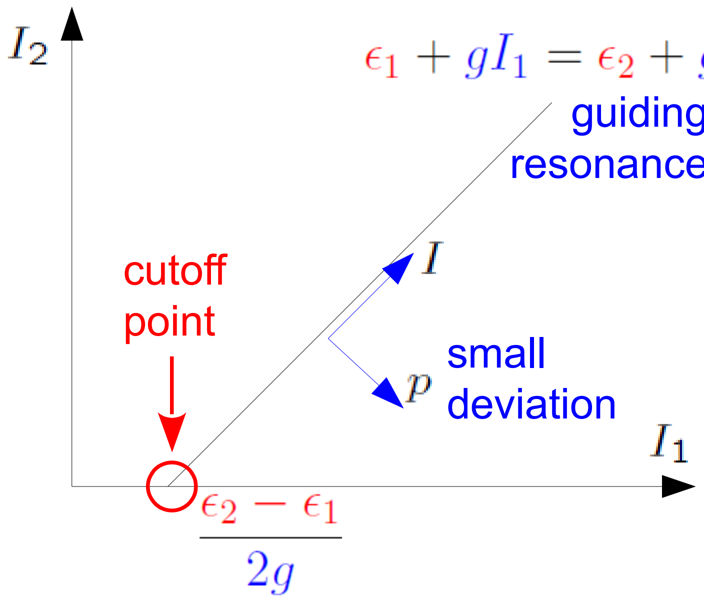
Chaos on three sites

Geometrical view of $(I_1, I_2) \to (I, p)$



Chaos on three sites

Geometrical view of $(I_1, I_2) \to (I, p)$



Strong disorder, weak nonlinearity:

$$gI_1, gI_2 \sim g\rho \ll W$$

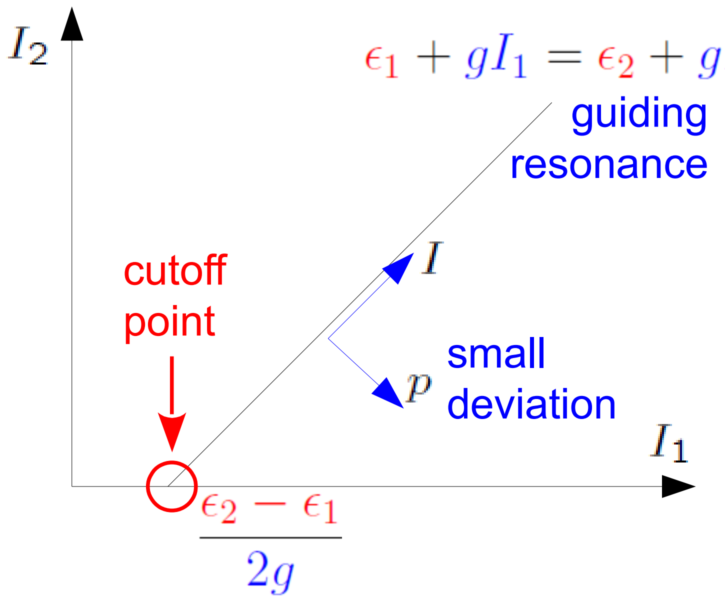
Need $|\epsilon_1 - \epsilon_2| \sim g\rho \ll W$

Look for a resonance

(guiding resonance)

Chaos on three sites

Geometrical view of $(I_1, I_2) \to (I, p)$



Strong disorder, weak nonlinearity:

$$gI_1, gI_2 \sim g\rho \ll W$$

Need $|\epsilon_1 - \epsilon_2| \sim g\rho \ll W$

Look for a resonance

(guiding resonance)

The pendulum frequency $\sim \sqrt{\Omega g\rho}$

The stochastic layer width $\propto \exp\left(-\frac{|\epsilon_2 - \epsilon_3|}{\sqrt{\Omega g\rho}}\right)$

Need $|\epsilon_2 - \epsilon_3| \sim \sqrt{\Omega g\rho} \ll W$

Look for another resonance

(layer resonance)

Chaos on more sites

$$-\Omega\left(\psi_1^*\psi_2+\psi_2^*\psi_1\right)+\epsilon_2\psi_2^*\psi_2-\Omega\left(\psi_2^*\psi_3+\psi_3^*\psi_2\right)$$

effective coupling 1 ↔ 3: $\dfrac{\Omega^2}{\epsilon_1-\epsilon_2}\left(\psi_1^*\psi_3+\psi_3^*\psi_1\right)$

works when $\epsilon_1 \approx \epsilon_3 \neq \epsilon_2$

Coupling + nonlinearity → effective couplings of the form

$$\psi_1^*\psi_2^*\psi_3^*\psi_4\psi_5\psi_6 \to \cos(\theta_1+\theta_2+\theta_3-\theta_4-\theta_5-\theta_6)$$

Guiding and layer resonances can be generated in high orders of the perturbation theory

Competition: number of combinations ↔ power of the coupling constants

Chaos comes from rare resonant triples: $w(\rho) \sim \min\left\{\left(\frac{\Omega}{W}\right)\left(\frac{g\rho}{W}\right),\left(\frac{g\rho}{W}\right)^2\right\}$

Basko, *Ann. Phys.* **326**, 1577 (2011)

A one-parameter model

$$H(p,q) = \sum_n \left(\frac{p_n^2}{2} + \frac{\omega_n^2 q_n^2}{2} \right) + \frac{1}{4} \sum_n (q_{n+1} - q_n)^4$$

disorder : $\frac{1}{2} \leq \omega_n^2 \leq \frac{3}{2}$

coupling + nonlinearity: $\epsilon = \frac{p_n^2}{2} + \frac{\omega_n^2 q_n^2}{2}$

Rescaled Hamiltonian of a triple close to resonance:

$$\frac{H + \text{const}}{(27/4)\epsilon^2/\omega^4} = \delta_{12}|\psi_1|^2 - (\delta_{12} + \delta_{23})|\psi_2|^2 + \delta_{23}|\psi_3|^2 + \\ + \frac{|\psi_1|^4}{2} + \frac{|\psi_1 - \psi_2|^4}{2} + \frac{|\psi_2 - \psi_3|^4}{2} + \frac{|\psi_3|^4}{2}$$

rescaled

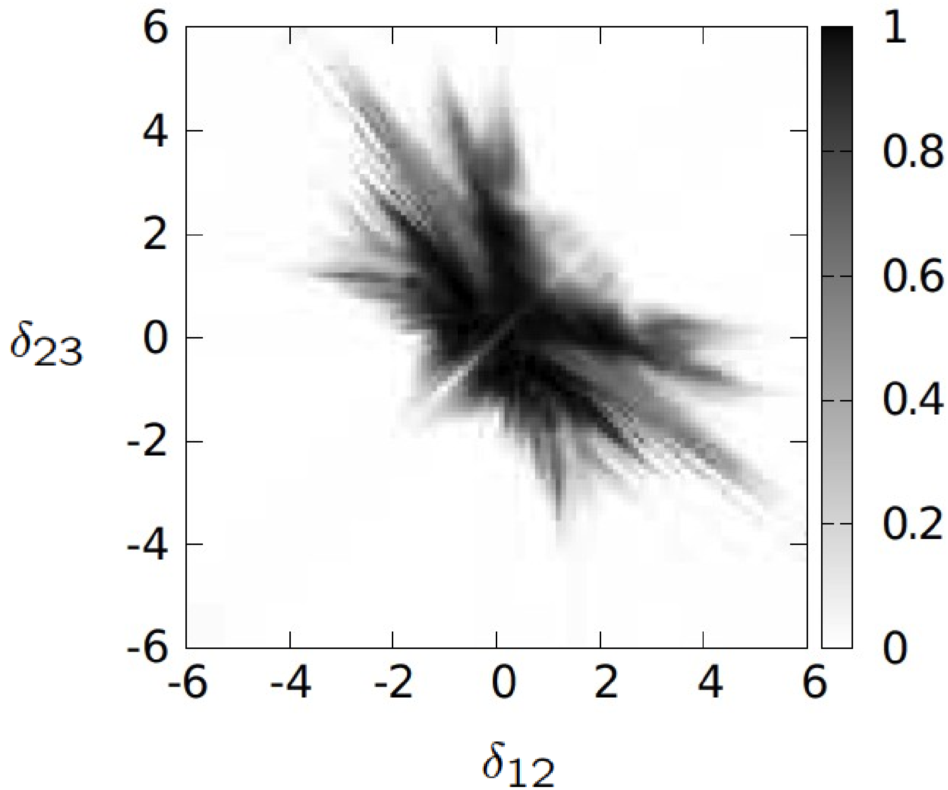
detunings: $-\infty < \delta_{12}, \delta_{23} < \infty$

initial condition: $|\psi_1|^2 = |\psi_2|^2 = |\psi_3|^2 = 1$

No parameters in the rescaled Hamiltonian ⇒ $w(\epsilon \to 0) = a\epsilon^2$

A numerical coefficient to be determined numerically

Numerics for the resonant triple



Phase-averaged probability of chaos (greyscale)

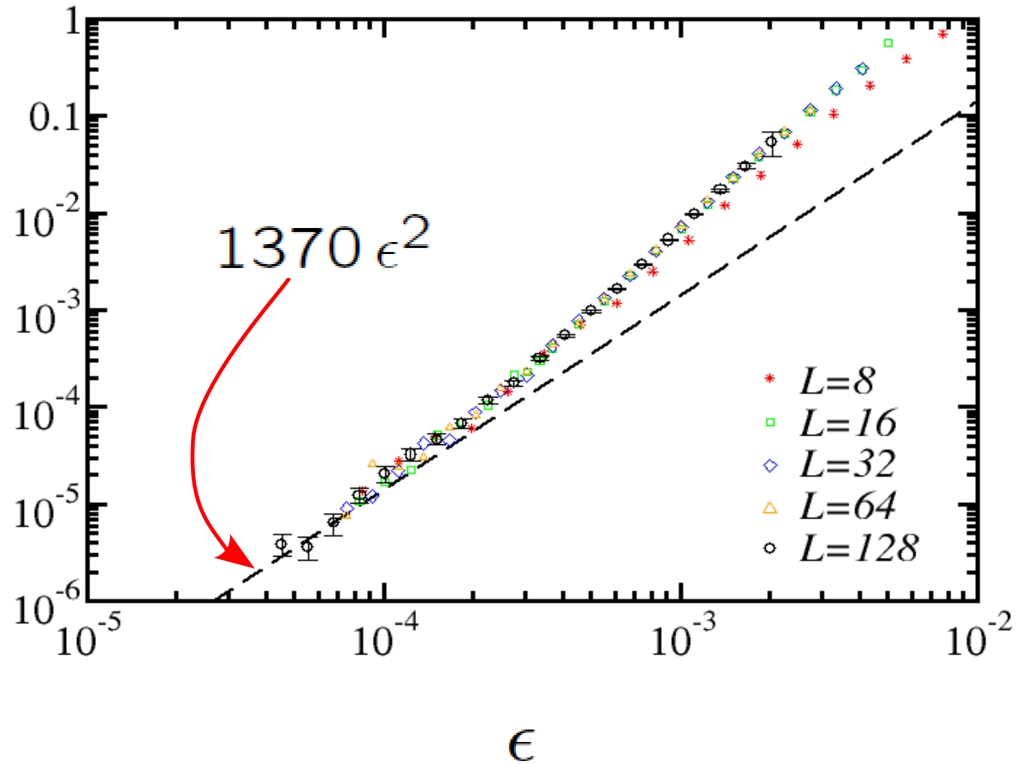
coefficient

$$w(\epsilon) \approx 1370\,\epsilon^2$$

Confinement in both directions:
a numerical proof that two resonances are needed for chaos

Numerics for the full chain

$$\frac{1}{L} \ln \frac{1}{1 - P_L}$$

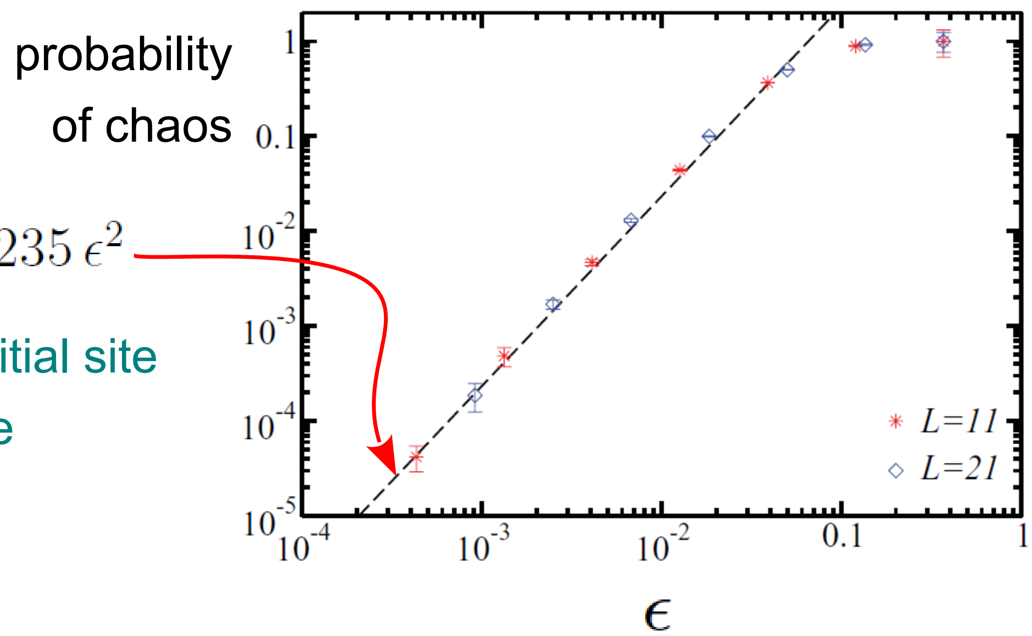


Single-site excitation

Initial excitation is concentrated on a single site in the middle of the chain

$$P_1(\epsilon) = 235\,\epsilon^2$$

The probability for the initial site to be on a resonant triple



Conclusions of Lecture 1

1. Anderson localization + weak nonlinearity → weak chaos

2. Such chaos is characterized by a probability

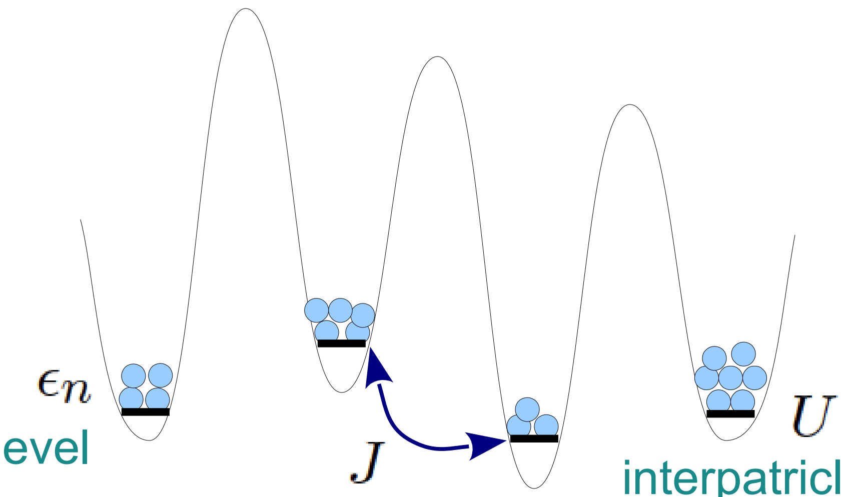
   $L \to \infty$, finite energy density ⇨ probability → 1

3. At weak nonlinearity chaos nucleates on rare local spots

Bosonic atoms in disordered optical lattices

Bose-Hubbard Hamiltonian:

$$\hat{H} = \sum_n \epsilon_n \hat{b}_n^\dagger \hat{b}_n + U \sum_n \hat{b}_n^\dagger \hat{b}_n^\dagger \hat{b}_n \hat{b}_n - J \sum_n \left(\hat{b}_{n+1}^\dagger \hat{b}_n + \hat{b}_n^\dagger \hat{b}_{n+1} \right)$$



level position

tunneling amplitude

interpatricle repulsion

bosonic operator

$$\hat{b}_n = \frac{\psi_n}{\sqrt{\hbar}} + \hat{\xi}_n$$

quantum fluctuation

condensate wave function

Billy *et al.*, *Nature* **453**, 891 (2008)

Roati *et al.*, *Nature* **453**, 895 (2008)

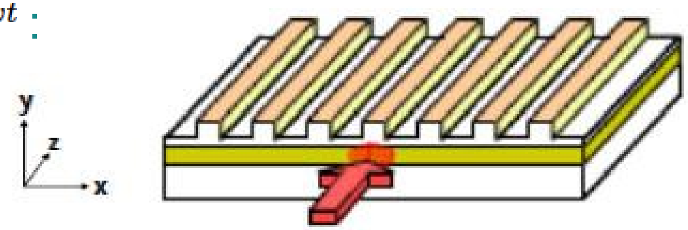
$$\frac{d\hat{b}_n}{dt} = \frac{i}{\hbar}\left[\hat{H}, \hat{b}_n\right] \quad \Rightarrow \quad i\hbar \frac{d\psi_n}{dt} = \epsilon_n \psi_n - J(\psi_{n-1} + \psi_{n+1}) + \frac{U}{\hbar}|\psi_n|^2 \psi_n$$

The classical limit of a bosonic field: $N_n \equiv \langle \hat{b}_n^\dagger \hat{b}_n \rangle \rightarrow \infty,\ \hbar \rightarrow 0,\ N_n \hbar \rightarrow |\psi_n|^2$

Light in evanescently coupled 1D waveguides

Wave equation for the electric field $E(\mathbf{r})e^{-i\omega t}$:

$$\varepsilon(\mathbf{r})\,\frac{\omega^2}{c^2}\,E + \nabla^2 E = \frac{\omega^2}{c^2}\,\chi^{(3)}|E|^2 E$$



Lahini *et al.*, *PRL* **100**, 013906 (2008)

Kerr nonlinearity

inhomogeneous dielectric structure → transverse eigenmodes

$$E(\mathbf{r}) = \sum_n \psi_n(z)\, u_\perp(x - x_n, y)\, e^{ikz}$$

$k \gg \partial/\partial x, \partial/\partial y$ paraxial approximation

discrete nonlinear Schrödinger equation

$$2ik\,\frac{\partial \psi_n}{\partial z} = q^2_{\perp,n}\psi_n + \dots$$

Anderson localization and nonlinearity

classical Hamiltonian

$$H(i\psi^*, \psi) = \sum_{n=-\infty}^{\infty} \omega_n \psi_n^* \psi_n + \sum_{n=-\infty}^{\infty} \frac{g}{2} \psi_n^* \psi_n^* \psi_n \psi_n - \sum_{n=-\infty}^{\infty} \Omega \left(\psi_n^* \psi_{n+1} + \psi_{n+1}^* \psi_n \right)$$

anharmonic oscillators

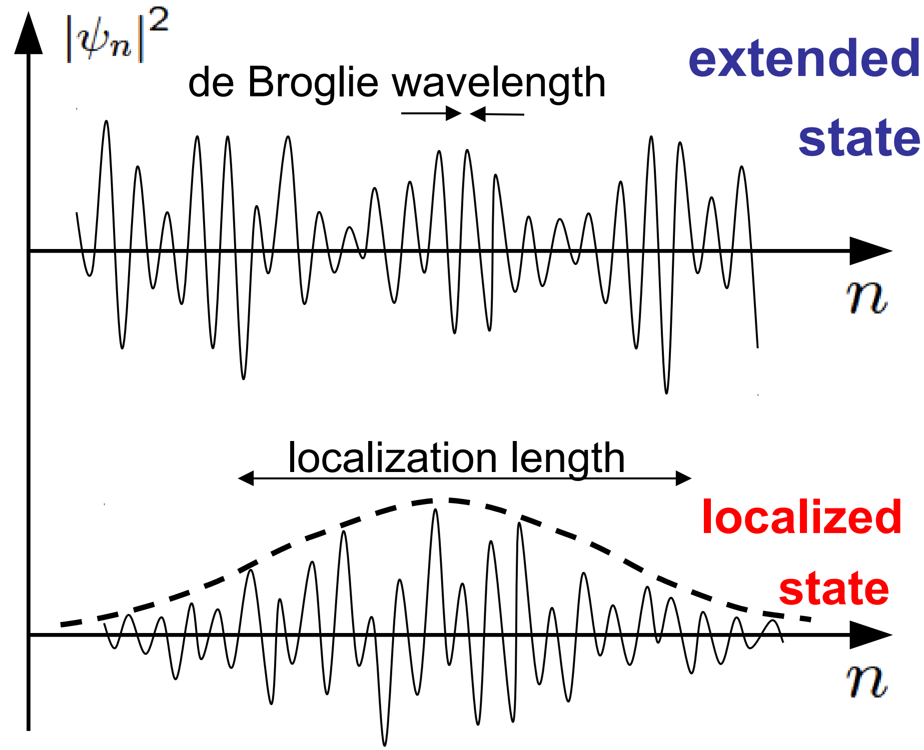
nearest-neighbor coupling

disorder → Anderson localization

$$i \frac{d\psi_n}{dt} = \omega_n \psi_n - \Omega \left(\psi_{n+1} + \psi_{n-1} \right) + g|\psi_n|^2 \psi_n$$

nonlinearity

In one dimension
all eigenstates are localized



Anderson localization and nonlinearity

$$i\,\frac{d\psi_n}{dt} = \omega_n\psi_n - \Omega(\psi_{n+1} + \psi_{n-1}) + g\psi_n^*\psi_n^2$$

WEAK coupling

↓

STRONG localization

WEAK

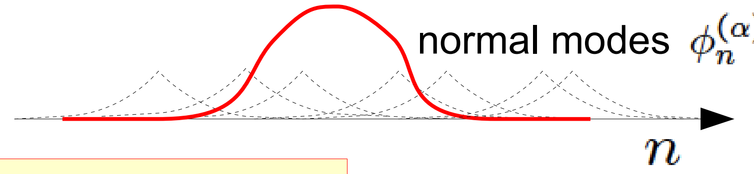
nonlinearity

What is the effect of a WEAK nonlinearity on STRONG Anderson localization?

The problem of wave packet spreading

<u>Linear system</u>: Anderson localization

$$\psi_n(t) = \sum_\alpha c_\alpha \phi_n^{(\alpha)} e^{-i\omega_\alpha t}, \qquad c_\alpha = \sum_n \phi_n^{(\alpha)} \psi_n(0)$$



The wave packet remains exponentially localized forever

<u>Nonlinear system</u>: interaction between the normal modes

$$i\,\frac{dc_\alpha}{dt} = \sum_{\beta\gamma\delta} V_{\alpha\beta\gamma\delta}\, e^{i(\omega_\alpha+\omega_\beta-i\omega_\gamma-i\omega_\delta)t}\, c_\beta^* c_\gamma c_\delta$$

→ small correction?

→ chaotic behavior?

Numerical integration: $\langle \Delta x^2 \rangle \propto t^p,\ p \sim 0.3 - 0.4$ subdiffusion

Shepelyansky (1993); Molina (1998); Kopidakis *et al.* (2008); Pikovsky & Shepelyansky (2008); Skokos *et al.* (2009); Skokos & Flach (2010); Laptyeva *et al.* (2010); Bodyfelt *et al.* (2011)

Indications for slowing down: Mulansky *et al.* (2011); Michaely & Fishman (2012)

KAM theorem, perturbation theory: $p \to 0$

Bourgain & Wang (2008); Wang & Zhang (2009); Fishman *et al.* (2009); Johansson *et al.* (2010)

Experiment: subdiffusion (non-universal exponent) Lucioni *et al.* (2011)

Other disordered nonlinear chains

$$H(p,q) = \sum_n \left[\frac{p_n^2}{2m} + U(q_n)\right] + \sum_{n,n'} \frac{m\Omega_{nn'}^2}{2} q_n q_{n'}$$

existence of localized solutions
corresponding to invariant tori

Fröhlich, Spencer & Wayne, *J. Stat. Phys.* **42**, 247 (1986)

(one conserved quantity;
two different terms responsible for anharmonicity and coupling, like in NLS)

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Classical spins, $|\vec{S}_n| = 1$: $H(\vec{S}) = \sum_n \left(\vec{h}_n \cdot \vec{S}_n + J\,\vec{S}_n \cdot \vec{S}_{n+1}\right)$

$$\{F, G\} = \sum_n \vec{S}_n \cdot \left[\frac{\partial F}{\partial \vec{S}_n} \times \frac{\partial G}{\partial \vec{S}_n}\right] \Rightarrow \frac{d\vec{S}_n}{dt} = \vec{S}_n \times \left(\vec{h}_n + J\vec{S}_{n-1} + J\vec{S}_{n+1}\right)$$

Energy transport → fast decay of time correlations on local spots of the chain

Oganesyan, Pal & Huse, *Phys. Rev. B* **80**, 115104 (2009)

(one conserved quantity; nonlinearity and coupling are governed by the same *J*)