Effect of nonlinearity on Anderson localization of classical waves

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Conclusions of Lecture 1

1. Anderson localization + weak nonlinearity → weak chaos

2. Such chaos is characterized by a probability

   $L \to \infty$, finite energy density ⇨ probability → 1

3. At weak nonlinearity chaos nucleates on rare local spots

How do disordered nonlinear chains behave in the chaotic regime?

Chaos and ergodicity

$$H_R(J_1, J_2, \phi_1, \phi_2) = J_1^2 + J_2^2 + J_1 J_2 - \cos(\phi_1) - \cos(\phi_2)$$

trajectory piercing the surface H_R = const, $\phi_1 = 0$ in the 4d phase space:

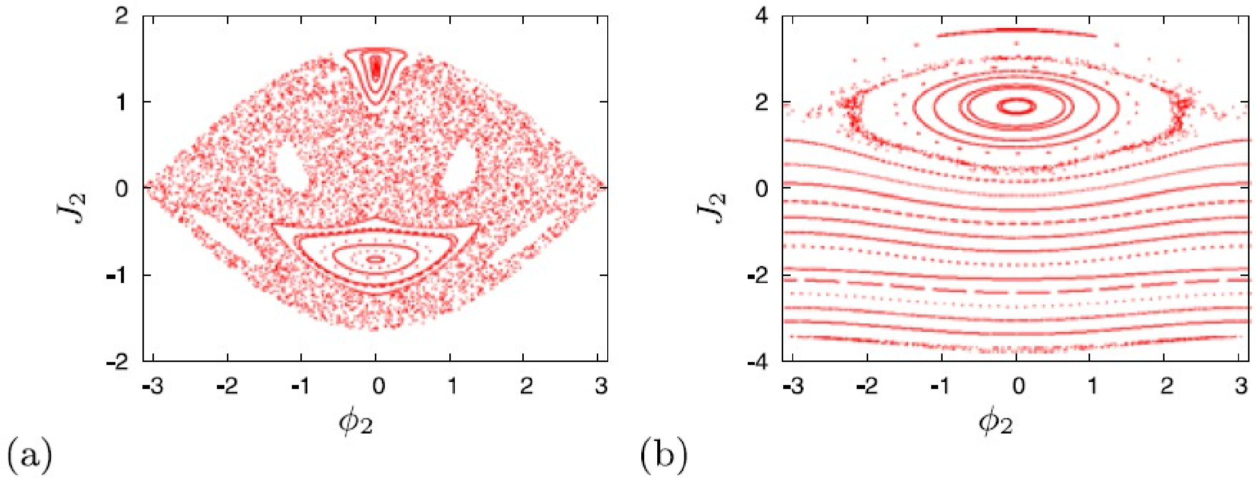

Fig. 11 Poincaré sections of variables ϕ_2, J_2 for the resonance Hamiltonian (12) at $\phi_1 = 0$. (**a**) $H_R = 0$, here chaos is dominant. (**b**) $H_R = 10$, here the dynamics is typically quasiperiodic

Mulansky, Ahnert, Pikovsky & Shepelyansky, *J. Stat. Phys.* **145**, 1256 (2011)

Chaos → probabilistic description

Microcanonical distribution

Probability to find the system in the vicinity of a point (**q**,**p**) in the phase space:

$$d\mathcal{P} = A(E)\,\delta(H(\mathbf{q},\mathbf{p}) - E)\,dq_1 \ldots dq_N\,dp_1 \ldots dp_N$$

normalization factor

assuming energy to be the only conserved quantity

phase space measure invariant under dynamics conservation of the phase volume by Liouville theorem

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entropy

$$\frac{1}{A(E)} = \int \delta(H(\mathbf{q},\mathbf{p}) - E)\,d\mathbf{q}\,d\mathbf{p} = \frac{1}{A_0}\,e^{S(E)}$$

unimportant constant to keep the dimensionality

$$\frac{dS(E)}{dE} = \beta \equiv \frac{1}{T}$$

temperature

System partitioning

$\mathbf{x} = (q_1, \ldots, q_{N_X}, p_1, \ldots, p_{N_X})$ $\mathbf{y} = (q_{N_X+1}, \ldots, q_N, p_{N_X+1}, \ldots, p_N)$

subsystem X	subsystem Y

$$N_X, N - N_X \gg 1$$

$$H(\mathbf{x}, \mathbf{y}) = H_X(\mathbf{x}) + H_{int}(\mathbf{x}, \mathbf{y}) + H_Y(\mathbf{y})$$

weak coupling

Local equilibration:

each subsystem explores

its own phase space

at fixed E_X and E_Y

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$$\begin{aligned} 1 &= A(E)\int \delta(H_X(\mathbf{x}) + H_Y(\mathbf{y}) - E)\, d\mathbf{x}\, d\mathbf{y} \\ &= A(E)\int dE_X \int d\mathbf{x}\, \delta(E_X - H_X) \int d\mathbf{y}\, \delta(E - E_X - H_Y) \\ &= \frac{A_0}{A_X A_Y}\int e^{S_X(E_X)+S_Y(E-E_X)-S(E)}\, dE_X \end{aligned}$$

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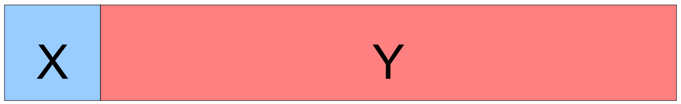
\end{aligned}
$$

the integral is determined by the vicinity of the maximum

$$\frac{dS_X(E_X)}{dE_X} + \frac{dS_Y(E - E_X)}{dE_X} = 0 \quad \Rightarrow \quad T_X = T_Y$$

Global equilibration: the subsystems exchange energy to make the temperatures equal

Canonical distribution



$N \gg N_X \gg 1:$ subsystem + reservoir

$$\mathcal{P}_X(\mathbf{x}) = A(E) \int \delta(H_X(\mathbf{x}) + H_Y(\mathbf{y}) - E)\, d\mathbf{y} \approx \frac{e^{-\beta H_X(\mathbf{x})}}{Z(\beta)}$$

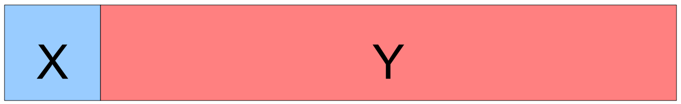
$$\beta = \frac{dS_Y}{dE_Y} \approx \frac{dS}{dE}$$

temperature determined by the reservoir

$$\langle E_X \rangle = \int H_X(\mathbf{x}) \frac{e^{-\beta H_X(\mathbf{x})}}{Z(\beta)}\, d\mathbf{x}$$

average energy

Canonical distribution



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average energy

microcanonical equation of state:

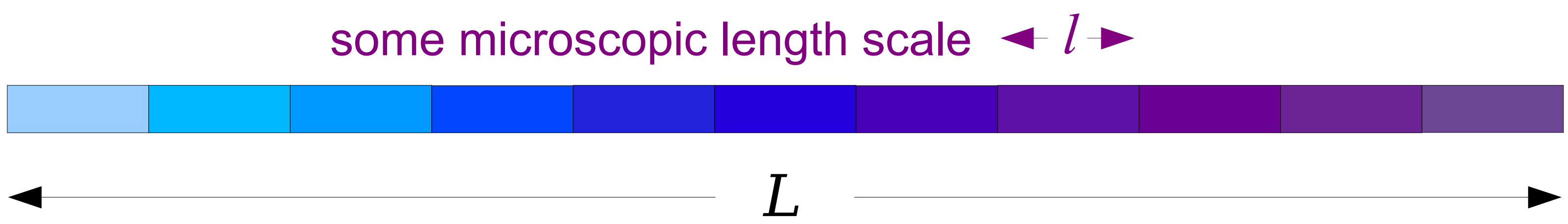
$$\beta(E) = \frac{\partial}{\partial E} \ln \int \delta(H(\mathbf{q}, \mathbf{p}) - E)\, d\mathbf{q}\, d\mathbf{p}$$

canonical equation of state:

$$E(\beta) = -\frac{\partial}{\partial \beta} \ln \int e^{-\beta H(\mathbf{q},\mathbf{p})}\, d\mathbf{q}\, d\mathbf{p}$$

In the thermodynamic limit **THEY ARE THE SAME**

Energy transport in extended systems



Local equilibration: each subsystem equilibrates at its own energy density

$$\frac{E_i}{l_i} = \mathcal{E}(x_i) \quad \Rightarrow \quad T(x_i)$$

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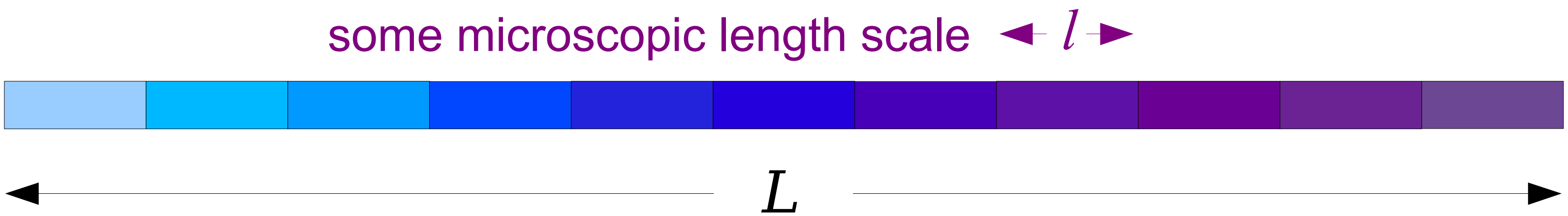

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Global equilibration: energy transport across the system

closed system of equations

$$\begin{cases} J = -\kappa(T)\,\dfrac{\partial T}{\partial x} & \text{energy current (from the microscopic dynamics)} \\[2ex] \dfrac{\partial \mathcal{E}}{\partial t} = -\dfrac{\partial J}{\partial x} & \text{continuity equation (energy conservation)} \\[2ex] \mathcal{E} = \mathcal{E}(T) & \text{thermodynamic equation of state (the same for all subsystems)} \end{cases}$$

$\kappa(T)$ – transport coefficient (thermal conductivity)

Several extensive conserved quantities

ensemble	probability distribution	thermodynamically conjugate variables
closed system: microcanonical	$\mathcal{P}(\mathbf{q},\mathbf{p}) = A(C_1,\ldots,C_K)\prod_{i=1}^{K}\delta(\mathcal{I}_i(\mathbf{q},\mathbf{p}) - C_i)$	$\gamma_i = -\frac{\partial \ln A}{\partial C_i}$
open system: grand canonical	$\mathcal{P}(\mathbf{q},\mathbf{p}) = \frac{1}{Z(\gamma_1,\ldots,\gamma_K)}\exp\left[-\sum_{i=1}^{K}\gamma_i\,\mathcal{I}_i(\mathbf{q},\mathbf{p})\right]$	$C_i = -\frac{\partial \ln Z}{\partial \gamma_i}$

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$$J_i = -\sum_{j=1}^{K}\mathcal{L}_{ij}(\gamma_1,\dots,\gamma_K)\,\frac{\partial\gamma_j}{\partial x}$$ conserved currents

$$\frac{\partial \mathcal{C}_i}{\partial t} = -\frac{\partial J_i}{\partial x}$$ continuity equations

$$C_i = L\,\mathcal{C}_i(\gamma_1,\dots,\gamma_K)$$ thermodynamic equations of state

$\mathcal{L}_{ij}(\gamma_1,\dots,\gamma_K)$ – $K\times K$ matrix of transport coefficients

$\mathcal{L}_{ij} = \mathcal{L}_{ji}$ Onsager symmetry

In a linear system with Anderson localization,

the transport coefficients are ZERO

In a nonlinear system,

they are well-defined, finite, and measurable

Disordered nonlinear wave equations

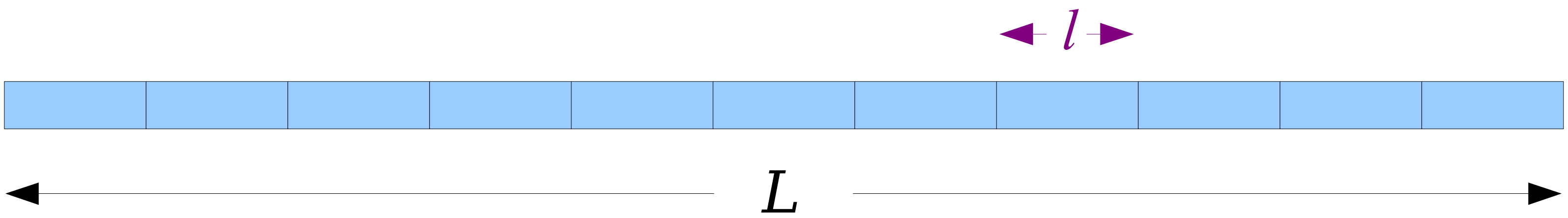
$$\frac{d^2u_n}{dt^2} = -\epsilon_n^2 u_n + \Omega^2 \left(u_{n+1} - 2u_n + u_{n-1}\right) - g u_n^3$$ energy only

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\Psi + g|\Psi|^2\Psi$$ energy & norm

$$i\frac{d\psi_n}{dt} = \epsilon_n \psi_n - \Omega\left(\psi_{n-1} + \psi_{n+1}\right) + g|\psi_n|^2\psi_n$$ energy & norm

Conserved total norm: $\mathcal{N} = \int |\Psi|^2\, dx$ or $\mathcal{N} = \sum_n |\psi_n|^2$

Effect of disorder on thermodynamics



Global equilibrium: T = const, but $\frac{E_i}{l_i} = \mathcal{E}(x_i)$ are different because of disorder

Total energy: $E = \sum_i E_i$

Spatially averaged energy density: $\mathcal{E} = \frac{E}{L} = \frac{1}{L} \sum_i l_i \, \mathcal{E}(x_i)$

Coarse-graining in space is equivalent to averaging over disorder

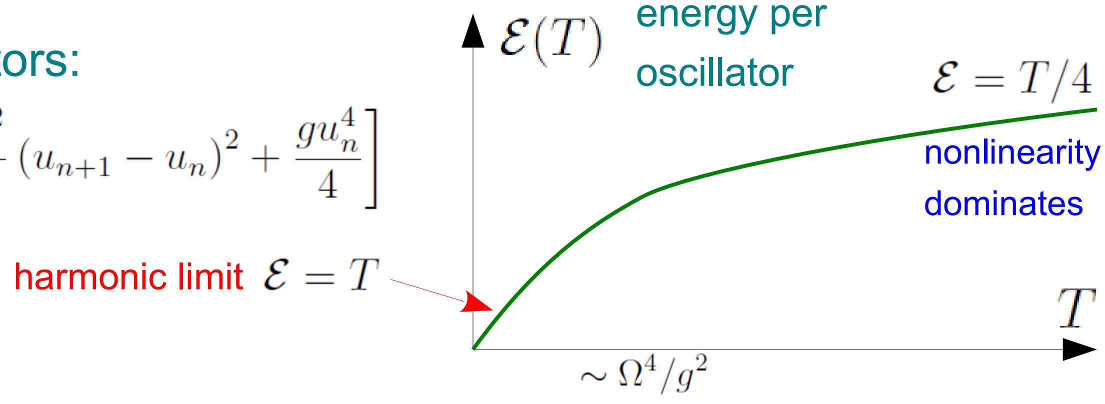
$$\mathcal{E} = -\frac{1}{L} \frac{\partial \ln Z(\beta)}{\partial \beta}$$

← the self-averaging quantity

Thermodynamics of nonlinear chains

1. Anharmonic oscillators:

$$H = \sum_n \left[\frac{p_n^2}{2} + \frac{\epsilon_n^2 u_n^2}{2} + \frac{\Omega^2}{2}(u_{n+1} - u_n)^2 + \frac{g u_n^4}{4} \right]$$



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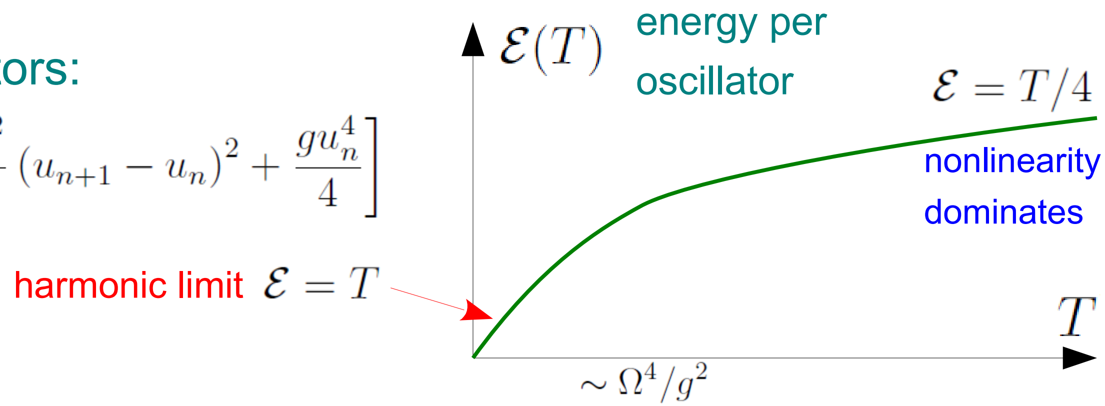
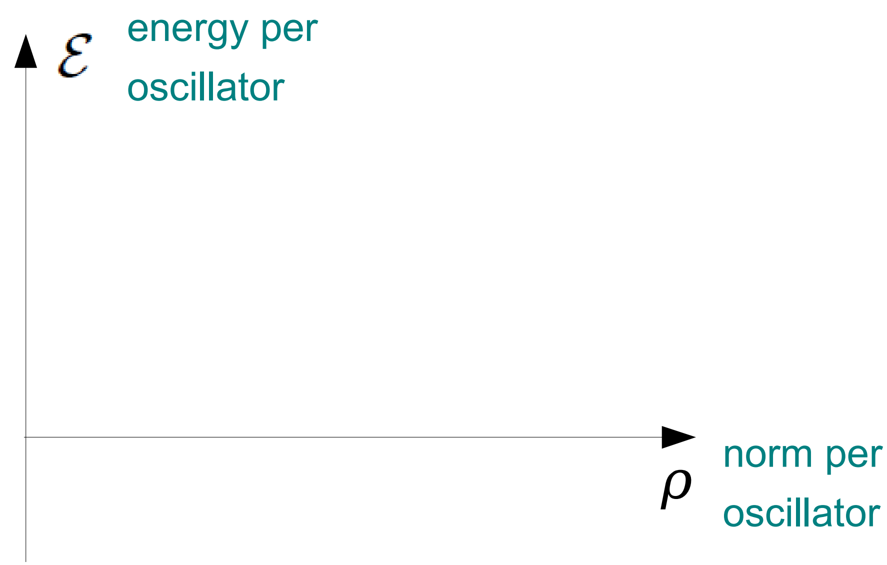


2. NLS chain: $$H = \sum_n \left[\epsilon_n \psi_n^* \psi_n - \Omega \left(\psi_n^* \psi_{n+1} + \psi_{n+1}^* \psi_n \right) + \frac{g}{2} \psi_n^* \psi_n^* \psi_n \psi_n \right]$$

$$\mathcal{N} = \sum_n |\psi_n|^2$$



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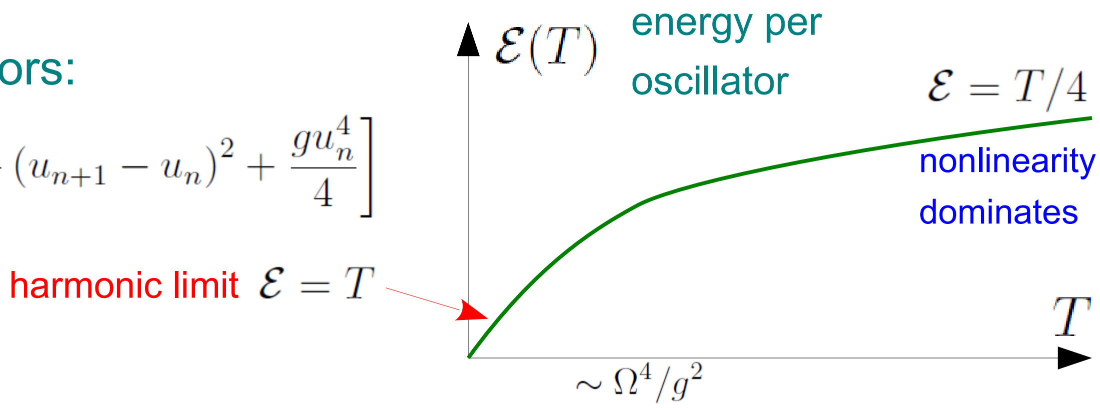

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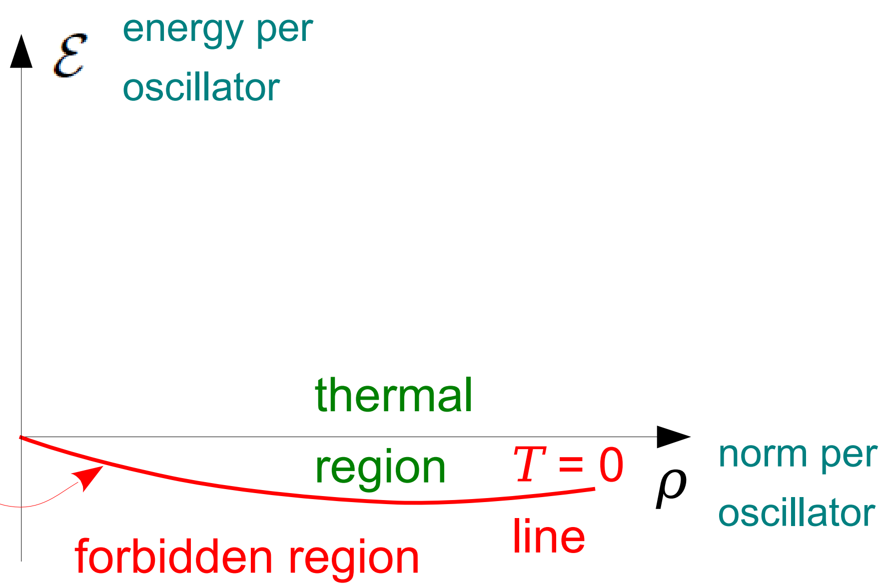

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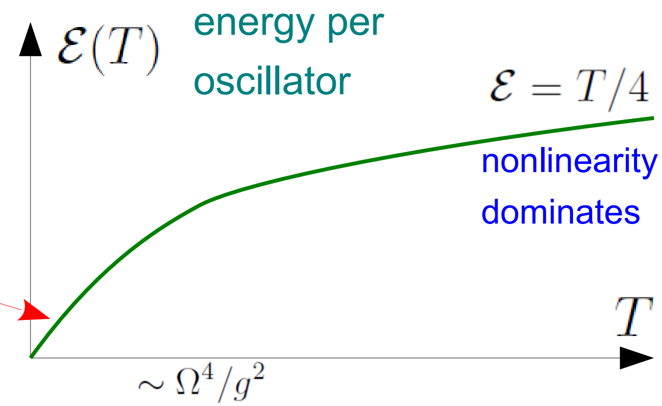

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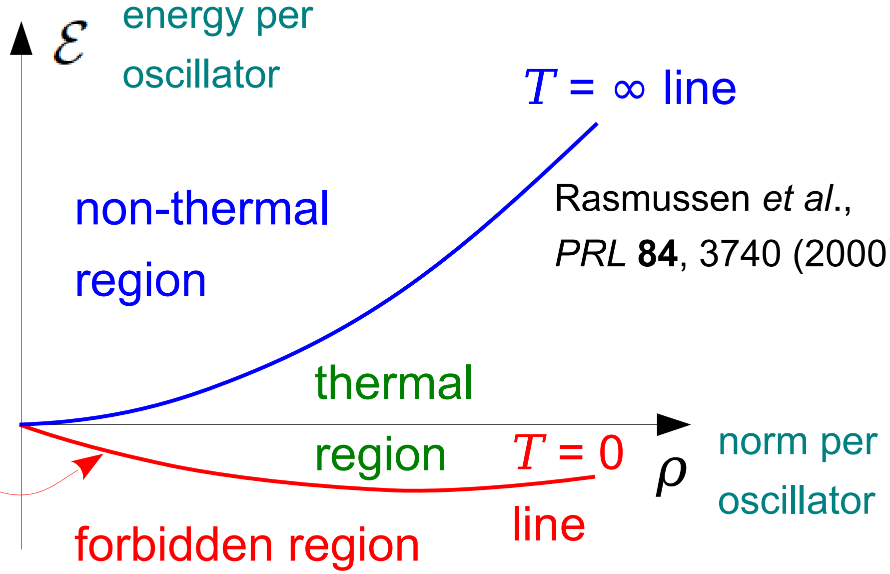

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Bose-Einstein condensation

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min H @ fixed $\mathcal{N}$ ⇨ $\frac{\partial}{\partial \psi_n^*} (H - \mu \mathcal{N}) = 0$

$$\epsilon_n \psi_n^\mu - \Omega(\psi_{n+1}^\mu + \psi_{n-1}^\mu) + g|\psi_n^\mu|^2 \psi_n^\mu = \mu \psi_n^\mu$$

GP equation for the condensate wave function

The solution is necessarily delocalized
otherwise not the ground state
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Linearization around the condensate:

$$\psi_n(t) = \psi_n^\mu e^{-i\mu t} + u_n e^{-i(\mu+\omega)t} + v_n e^{-i(\mu-\omega)t}$$

Bogolyubov modes

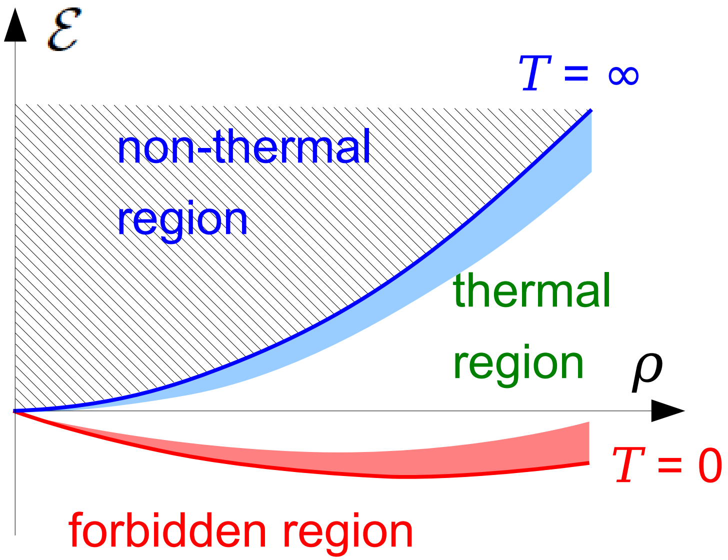
are all localized, but the localization length diverges ~ $1/\omega^2$
typical for Goldstone modes

Bilas & Pavloff, *Eur. Phys. J. D* **40**, 387 (2006); Ziman, *PRL* **49**, 337 (1982)

Transport in disordered nonlinear chains

$$H = \sum_n \left[\epsilon_n \psi_n^* \psi_n - \Omega \left(\psi_n^* \psi_{n+1} + \psi_{n+1}^* \psi_n \right) + \frac{g}{2} \psi_n^* \psi_n^* \psi_n \psi_n \right] \qquad \mathcal{N} = \sum_n |\psi_n|^2$$

1. Low temperatures
   transport coefficients diverge at $T \to 0$ (**?**)
   approaching superfluidity
2. High temperatures
   behavior at small ρ to be discussed later
3. Non-thermal region
   ??? how to define transport coefficients?



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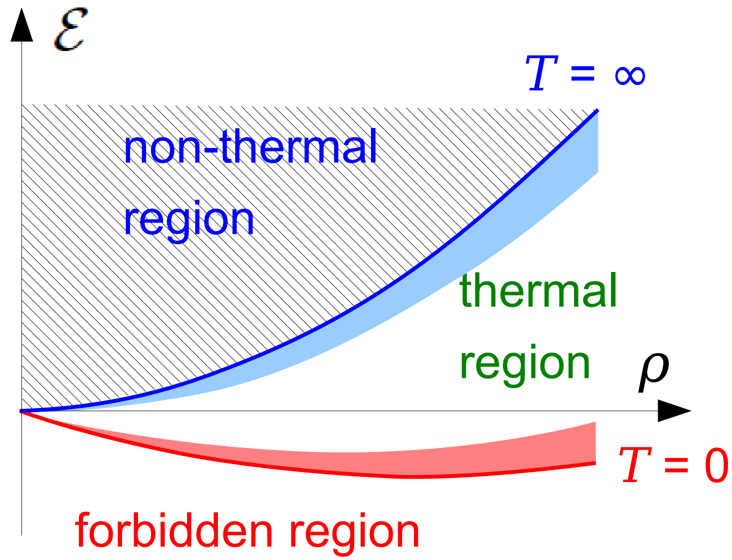

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Behavior of thermal conductivity as a function of T
seems analogous to conductivity of NLS chain at $T \to \infty$ as a function of ρ (**?**)

NLS chain @ high temperature & low norm density

$$\mathcal{P} \propto e^{-\beta H - \gamma \mathcal{N}}$$

small (β) large (γ)

Equations of state:

$$\rho = 1/\gamma + O(\beta)$$
$$\mathcal{E} = g/\gamma^2 + O(\beta)$$

Currents:

$$J^{\varepsilon} = \mathcal{L}_{11}\partial_x\beta + \mathcal{L}_{12}\partial_x\gamma$$
$$J^{\rho} = \mathcal{L}_{21}\partial_x\beta + \boxed{\mathcal{L}_{22}\partial_x\gamma}$$

Eliminate β, γ, $\mathcal{E}$ from the transport equations:

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial x} D(\rho) \frac{\partial\rho}{\partial x}$$

Anharmonic oscillator chain @ low temperature

$$\mathcal{P} \propto e^{-\beta H}$$

large (β)

$$\mathcal{E} = 1/\beta + O(1/\beta^2)$$

$$J^{\varepsilon} = \mathcal{L}_{11}\partial_x\beta$$

Eliminate β from the transport equations:

$$\frac{\partial\mathcal{E}}{\partial t} = \frac{\partial}{\partial x} D_{\varepsilon}(\mathcal{E}) \frac{\partial\mathcal{E}}{\partial x}$$

$D(\rho)$ and $D_{\varepsilon}(\mathcal{E})$ have similar density dependence

Flach, Krimer & Skokos
PRL **102**, 024101 (2009)

$$i\,\frac{d\psi_n}{dt} = \omega_n \psi_n - \Omega(\psi_{n+1} + \psi_{n-1}) + g\psi_n^* \psi_n^2$$

1. Strong disorder
2. Weak nonlinearity
3. High temperature

} → worst conditions for transport

4. (Almost) arbitrary initial condition with extensive norm and energy

- Chaos is concentrated on rare local spots
- They produce stochastic Langevin-like force
- It redistributes energy between sites
- Force statistics → transport coefficients

$$D(\rho) \sim \exp\left(-C \ln^3 \frac{W}{g\rho}\right)$$

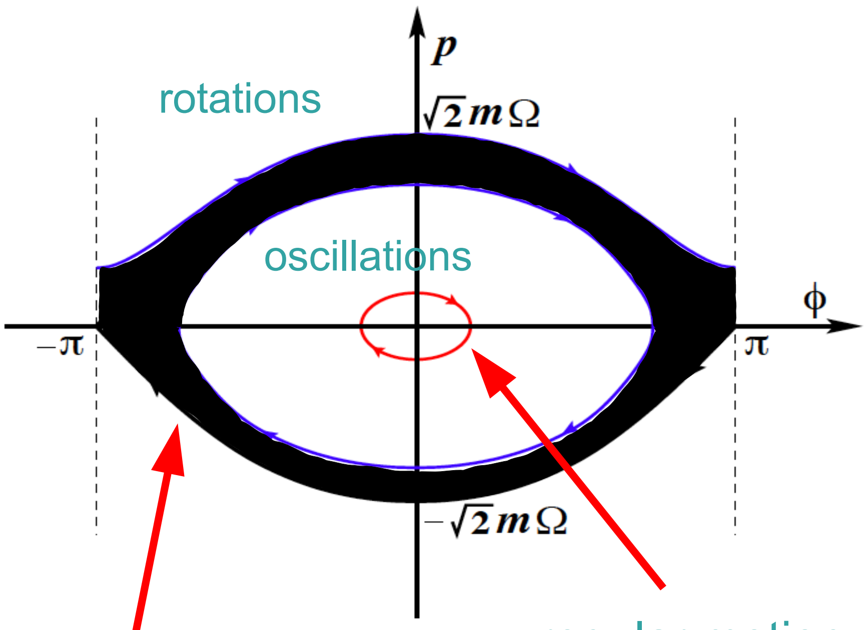
(W: disorder; $g\rho$: nonlinearity)

D. M. Basko, *Ann. Phys.* **326**, 1577 (2011)

$$g\rho \ll \Omega \ll W$$

Perturbed pendulum:

$$H(p,\phi,t) = \frac{p^2}{2m} - m\Omega^2 \cos\phi - V\cos(\phi - \omega t)$$



ergodic trajectories
within
the stochastic layer

regular motion
survives

Stochastic layer area:

$$W_s \equiv \int_{\text{layer}} \frac{dp\, d\phi}{2\pi} \sim \frac{V}{\Omega} e^{-|\omega|/\Omega}$$

Melnikov-Arnold integral

$$|\omega| \gg \Omega$$

Continuous spectrum
of the chaotic motion:

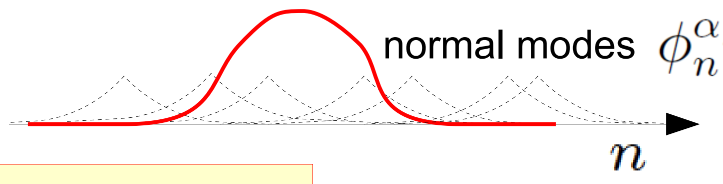
$$\left\langle e^{i\phi(t)} e^{-i\phi(t')} \right\rangle_\omega \sim \frac{1}{\Omega} e^{-|\omega|/\Omega}$$

review: B. Chirikov (1979)

The problem of wave packet spreading

Linear system: Anderson localization

$$\psi_n(t) = \sum_\alpha c_\alpha \phi_n^\alpha e^{-i\omega_\alpha t}, \qquad c_\alpha = \sum_n \phi_n^\alpha \psi_n(0)$$



The wave packet remains exponentially localized forever

Nonlinear system: interaction between the normal modes

$$i\frac{dc_\alpha}{dt} = \sum_{\beta\gamma\delta} V_{\alpha\beta\gamma\delta}\, e^{i(\omega_\alpha+\omega_\beta-\omega_\gamma-\omega_\delta)t}\, c_\beta^* c_\gamma c_\delta$$

→ small correction?

→ chaotic behavior?

$$V_{\alpha\beta\gamma\delta} = g\sum_n \phi_n^\alpha \phi_n^\beta \phi_n^\gamma \phi_n^\delta$$

The wave packet can still remain localized, with some probability

If the wave packet is chaotic, how does it expand?

The problem of wave packet spreading

Numerical integration: $\langle \Delta x^2 \rangle \propto t^p, \; p < 1$: **subdiffusion**

p = 2/5 Shepelyansky (1993)

p = 0.27 Molina (1998)

p = 0.3 – 0.4 Kopidakis, Komineas, Flach & Aubry (2008); Pikovsky & Shepelyansky (2008)

p = 1/3 Flach, Krimer & Skokos (2009); Skokos, Krimer, Komineas & Flach (2010)

crossover p = 1/2 → p = 1/3 Laptyeva, Bodyfelt, Krimer, Skokos & Flach (2010, 2011)

higher density *lower density* crossover from strong to weak chaos

Indications for slowing down: Mulansky, Ahnert & Pikovsky (2011); Pikovsky & Fishman (2011); Michaely & Fishman (2012)

KAM theorem, perturbation theory: $p \to 0$

Bourgain & Wang (2008); Wang & Zhang (2009); Fishman *et al.* (2009); Johansson *et al.* (2010)

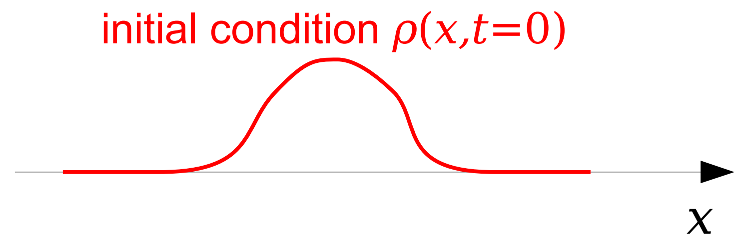
Experiment (cold atoms in the Aubry-Andre potential): subdiffusion with a non-universal exponent

Lucioni *et al.* (2011)

Nonlinear diffusion equation

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} D(\rho) \frac{\partial \rho}{\partial x}$$

density-dependent
diffusion coefficient



$$D(\rho) = D_0 \rho^a$$ yielding $\langle \Delta x^2 \rangle \propto t^{2/(a+2)}$ Mulansky & Pikovsky (2010)

$$D(\rho) = C_1 \frac{g^2 \rho^2}{\Omega} \left(1 - e^{-C_2 g \rho / \Omega}\right)^2$$ interpolate between ρ^2 and ρ^4

Flach (2010); Laptyeva, Bodyfelt & Flach (2013)

$$D(\rho) \sim \exp\left(-C \ln^3 \frac{W}{g\rho}\right)$$

local structure of chaos
strong disorder & low density Basko (2011)

gives $\langle \Delta x^2 \rangle \sim \exp\left(\ln^{1/3} t\right)$

slower than any power law

Power-law subdiffusion: an intermediate asymptotics?

$$D(\rho) = D_0 \rho^2, \ \langle \Delta x^2 \rangle \propto t^{1/2}$$

can be derived

Basko, PRE **89**, 022921 (2014)

Weak disorder + homogeneous chaos

$$i\frac{d\psi_n}{dt} = \epsilon_n\psi_n - \Omega(\psi_{n+1} + \psi_{n-1}) + g\psi_n^*\psi_n^2$$

disorder — nearest-neighbor coupling — anharmonicity

$$-\frac{W}{2} \le \epsilon_n \le \frac{W}{2}$$

1. Weak disorder: $\xi \approx 100\frac{\Omega^2}{W^2} \gg 1$ — each normal mode overlaps with many others

2. Moderate nonlinearity: $0.008\left(\frac{W}{\Omega}\right)^{7/2} \ll \frac{g\rho}{\Omega} \ll 0.13\left(\frac{W}{\Omega}\right)^{3/2}$

- nonlinearity strong enough → most of normal modes are chaotic
- nonlinearity weak enough → normal modes are still resolved

3. Infinite temperature: $T \to \infty, \quad \mu \to -\infty, \quad \frac{\mu}{T} = -\frac{1}{\rho}$

- suppress superfluid effects
- suppress thermoelectric phenomena

From microscopic to macroscopic

Kinetic equation for localized normal mode intensities

↓

Local thermalization

↓

Macroscopic transport coefficients (norm + energy)

↓

Eliminate the energy density at high temperatures

Nonlinear diffusion coefficient

$$D(\rho) = D_0 \rho^2$$

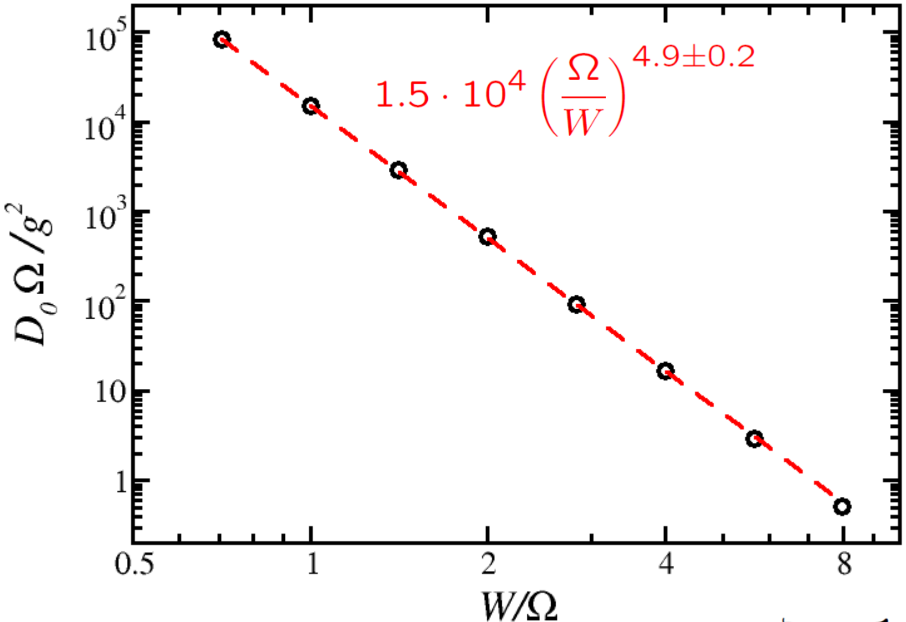
$$D_0 = \frac{\pi}{L} \sum_{\alpha,\beta,\gamma,\delta} (X_\alpha + X_\beta - X_\gamma - X_\delta)^2 V^2_{\alpha\beta\gamma\delta}\, \delta(\omega_\alpha + \omega_\beta - \omega_\gamma - \omega_\delta)$$

$\frac{\pi}{L} \to \infty$

$$X_\alpha = \sum_n n(\phi_n^\alpha)^2$$

$$V_{\alpha\beta\gamma\delta} = g \sum_n \phi_n^\alpha \phi_n^\beta \phi_n^\gamma \phi_n^\delta$$

δ: width $\to$ 0 *only after* $L \to \infty$



$\xi \approx 200$ $\xi \approx 1.5$

numerical evaluation

Nonlinear diffusion coefficient

$$D(\rho) = D_0 \rho^2$$

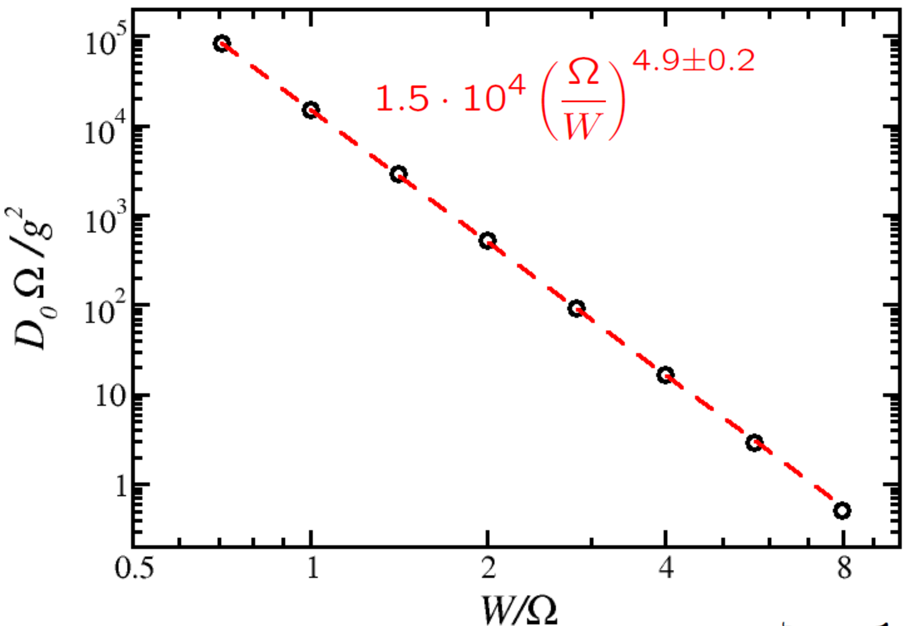
$$D_0 = \frac{\pi}{L} \sum_{\alpha,\beta,\gamma,\delta} (X_\alpha + X_\beta - X_\gamma - X_\delta)^2 V_{\alpha\beta\gamma\delta}^2 \, \delta(\omega_\alpha + \omega_\beta - \omega_\gamma - \omega_\delta)$$

$\frac{\pi}{L} \to \infty$

$\delta(\ldots)$: width → 0 *only after* $L \to \infty$

$$X_\alpha = \sum_n n (\phi_n^\alpha)^2$$

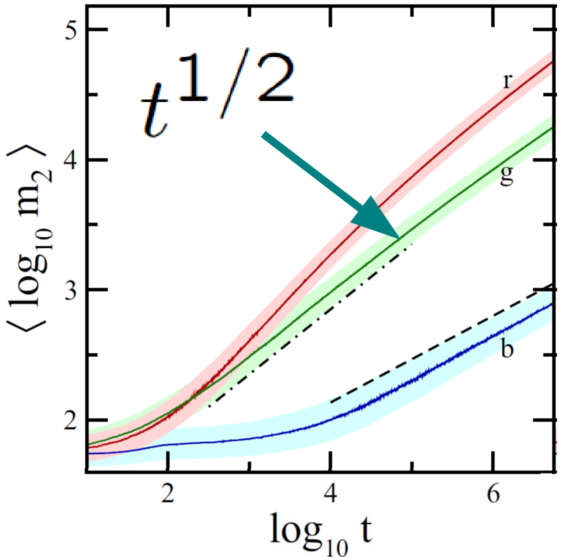
$$V_{\alpha\beta\gamma\delta} = g \sum_n \phi_n^\alpha \phi_n^\beta \phi_n^\gamma \phi_n^\delta$$



$\xi \approx 200$ $\xi \approx 1.5$

numerical evaluation

Comparison to numerics



Laptyeva, Bodyfelt, Krimer, Skokos & Flach,
EPL **91**, 30001 (2010)

$$\frac{W}{\Omega} = 4, \quad \frac{g}{\Omega}\sum_n |\psi_n(0)|^2 \approx 15$$

$$\Rightarrow \quad D_0 \approx 4$$

the present theory: $D_0 \approx 16$

Power-law subdiffusion: an intermediate asymptotics?

$$D(\rho) = D_0\rho^2,\ \langle \Delta x^2 \rangle \propto t^{1/2}$$

can be derived

D. M. Basko, PRE **89**, 022921 (2014)

slows down when ρ drops below $\left(10^{-2} - 10^{-3}\right) \times \left(\frac{W}{\Omega}\right)^{7/2}$

$$D(\rho) = D_0\rho^4,\ \langle \Delta x^2 \rangle \propto t^{1/3}$$

observed numerically at long times

No systematic derivation is available

The mechanism is unknown

When should it slow down?

Conclusions

1. **In the chaotic regime, disordered nonlinear chains obey the basic principles of statistical physics**

2. **Anderson localization manifests itself in vanishing transport coefficients at low densities**

3. **The long-time behavior of a finite-norm wave packet is still debated**