

Speckle fluctuations resolve the interdistance between incoherent point sources in complex media

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(Dated: July 22, 2014)

We study the fluctuations of the light emitted by two identical incoherent point sources in a disordered environment. The intensity-intensity correlation function and the speckle contrast, obtained after proper temporal and configurational averaging, encode the relative distance between the two sources. This suggests the intriguing possibility that intensity measurements at only one point in a speckle pattern produced by two incoherent sources can provide information about the relative distance between the sources, with a precision that is not limited by diffraction. The theory also suggests an alternative approach to Green function retrieval technique, where the correlations of the isotropic ambient noise detected by two receivers are replaced by a measurement at a single point of the noise due to two fluctuating incoherent sources.

PACS numbers: 42.25.Dd, 42.30.Ms, 05.40.-a

I. INTRODUCTION

Pushing the resolution limits of light microscopy, and understanding optical phenomena on scales below the diffraction limit, has been the driving force of what is known today as nano-optics [1]. To overcome this limit, most of the early work was focused on near-field optical microscopy and related techniques [2]. However, in recent years, new concepts in fluorescence microscopy have pushed the resolution of far-field imaging down to the nanometer range [3]. Most of these methods [4] rely on the accurate localization of individual fluorescent markers, that are isolated from one another on the basis of one or more distinguishing optical characteristics, or by selective or random activation of a bright and a dark state [3]. Determining the location of an isolated fluorescent marker is only limited by photon noise, and not by the diffraction barrier.

A key issue affecting these subwavelength imaging methods is the optical transparency of the media surrounding the light emitters. Taking advantage of the transparency of cells, fluorescence microscopy uniquely provides noninvasive imaging of the interior of cells and allows the detection of specific cellular constituents through fluorescence tagging. However, certain biological tissues or soft-matter systems (such as foams or colloidal suspensions) look turbid due to intense scattering of photons traveling through them [5]. The image formed

at a given point in the observation plane consists in a superposition of multiple fields, each arising from a different scattering sequence in the medium. This gives rise to a chaotic intensity distribution with numerous bright and dark spots known as a speckle pattern, producing a blurred image carrying no apparent information about the source position [6].

Techniques to measure the distance between individual nano-objects without actually imaging their position exist [7], Fluorescence Resonance Energy Transfer (FRET) being the most widespread example [8]. It relies on the near-field energy transfer between two fluorophores (donor and acceptor) emitting at different wavelengths. The FRET signal (*e.g.* the ratio between the intensities emitted by the donor and the acceptor at different wavelengths) depends on the donor-acceptor distance in the range $2 \sim 10$ nm. As such, it is not very sensitive to scattering problems. However, determining distances between two emitters in the range of 10 to 250 nm (of interest for example in the study of conformational changes in biomolecules) in a scattering medium still remains a challenging problem, not accessible either by fluorescence microscopy or FRET techniques.

Multiple scattering of waves had long been considered to be an unavoidable nuisance. However, contrary to intuition, there is an increasing number of examples showing that multiple scattering can actually enhance the performance of several applications based on wave propagation [9]: The enhanced spatio-temporal focusing by time reversal [10, 11], wavefront shaping of multiply scattered waves [12], the enhancement of the information capacity of telecommunication channels [13], or Green function retrieval from diffuse waves [14] are just a few illustrations.

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In this paper, we address the problem of two identical incoherent point sources emitting light simultaneously through a surrounding disordered environment. We calculate the fluctuations of the total emitted power and of the intensity measured in a given direction, or equivalently at a single point in the far field. We show that the intensity-intensity correlation function and the speckle contrast, obtained after proper temporal and configurational averaging, encode the relative distance between the two sources. This suggests that intensity measurements at only one point in a speckle pattern produced by two incoherent sources can be used to monitor the relative distance between the sources, as schematically illustrated in Fig. 1. The precision in the determination of the interdistance is in principle not limited by diffraction, but by the microstructure of the scattering medium.

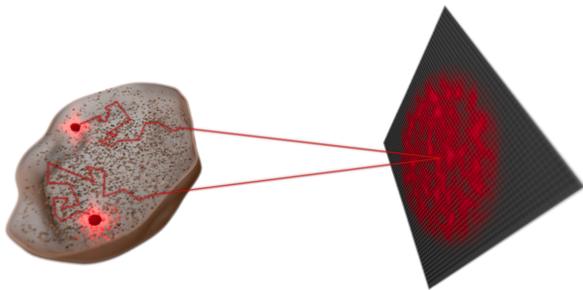


FIG. 1: The intensity radiated by two incoherent point sources in a complex medium form a speckle pattern that fluctuates in both space and time. The speckle fluctuations encode the relative distance between the sources.

II. FLUCTUATIONS IN THE POWER EMITTED BY TWO INCOHERENT SOURCES

We consider two point sources of light (electric dipoles) located at \mathbf{r}_1 and \mathbf{r}_2 in a disordered medium. The sources are characterized by their electric dipole moments $\mathbf{p}_1(t)$ and $\mathbf{p}_2(t)$, that are fluctuating quantities of the form $\mathbf{p}_k(t) = p_k(t)\mathbf{u}_k \exp(-i\omega t)$ with $p_k(t)$ a slowly varying random complex envelope and \mathbf{u}_k a unit vector defining the orientation of the dipole moment. This corresponds to a classical model for a quasi-monochromatic temporally incoherence source, such as a fluorescent source emitting at frequency ω . We assume that the two sources are uncorrelated (or mutually incoherent), so that $\overline{p_1(t)p_2^*(t)} = 0$, where the bar denotes averaging over the fluctuations of the sources. Using the (dyadic) Green function $\mathbf{G}(\mathbf{r}, \mathbf{r}', \omega)$ of the disordered medium, the electric fields at any point \mathbf{r} can be written :

$$\mathbf{E}(\mathbf{r}) = \mu_0\omega^2\mathbf{G}(\mathbf{r}, \mathbf{r}_1, \omega)\mathbf{p}_1 + \mu_0\omega^2\mathbf{G}(\mathbf{r}, \mathbf{r}_2, \omega)\mathbf{p}_2. \quad (1)$$

The intensity associated to this field is a time fluctuating and spatially varying quantity that forms a time-dependent speckle pattern.

Let us first consider the total power P emitted by the two sources. It reads

$$P = \int_S \frac{\epsilon_0 c}{2} |\mathbf{E}(\mathbf{r})|^2 dS \quad (2)$$

where S is a sphere with radius $R \rightarrow \infty$ that encloses the disordered medium and c is the speed of light in vacuum. For a *non absorbing* medium, the following relation can be derived from the vector form of Green's second identity [15] (the frequency dependence in the Green function is dropped for simplicity):

$$\frac{\omega}{c} \int_S \mathbf{G}(\mathbf{r}, \mathbf{r}_1)\mathbf{p}_1 \cdot \mathbf{G}^*(\mathbf{r}, \mathbf{r}_2)\mathbf{p}_2^* dS = \mathbf{p}_1 \cdot \text{Im}[\mathbf{G}(\mathbf{r}_1, \mathbf{r}_2)]\mathbf{p}_2^*. \quad (3)$$

From Eqs. (1-3), we obtain

$$P = \frac{\mu_0 \omega^3}{2} \sum_{j,j'=1}^2 p_j p_{j'}^* \text{Im}G_{jj'} \quad (4)$$

where the notation $\text{Im}G_{jj'} = \mathbf{u}_j \cdot \text{Im}[\mathbf{G}(\mathbf{r}_j, \mathbf{r}_{j'}, \omega)]\mathbf{u}_{j'}$ has been introduced for the sake of simplicity.

We first assume that a temporal averaging over the fluctuations of the sources can be performed, in one configuration of the disordered medium (frozen disorder). The averaged power \overline{P} is simply the sum of the averaged power emitted by each source independently, since the terms with $j \neq j'$ in Eq. (4) vanish upon time averaging. It reads as

$$\overline{P} = \frac{\pi\omega^2}{4\epsilon_0} (|p_1|^2 \rho_{11} + |p_2|^2 \rho_{22}) \quad (5)$$

where $\rho_{jj} = 2\omega/(\pi c^2)\text{Im}G_{jj}$ is the electric part of the local density of states (LDOS) at point \mathbf{r}_j [16]. However, a cross-term survives in the fluctuations of the total emitted power. Indeed, calculating the variance of P from Eq. (4), one obtains

$$\overline{P^2} - \overline{P}^2 = \frac{\mu_0^2 \omega^6}{4} [2|p_1|^2|p_2|^2 (\text{Im}G_{12})^2]. \quad (6)$$

The imaginary part of the two-point Green function $\text{Im}G_{12}$ in Eq. (6) is known to enter the expression of field-field spatial correlations in random fields, such as black-body radiation or volume speckle patterns [17, 18], the description of time-reversed fields [11], and is at the core of Green function retrieval techniques based on ambient noise correlations [14]. It is proportional to the cross density of states (CDOS), that was introduced in a different context for the description of spatial coherence in complex systems [19]. Physically, the CDOS counts the number of photonic eigenmodes connecting two points (in our case the source points) at a given frequency [20]. More precisely, the CDOS connecting \mathbf{r}_k to \mathbf{r}_j is given by $\rho_{jk} = 2\omega/(\pi c^2)\text{Im}G_{jk}$ [21]. Using the CDOS, and assuming that the two sources have the same amplitude

($p_1 = p_2 \equiv p$), the variance of the total emitted power can be rewritten as

$$\overline{P^2} - \bar{P}^2 = \frac{\pi^2 \omega^4}{8 \epsilon_0^2} |p|^4 \rho_{12}^2. \quad (7)$$

This equation is the first result in this paper. It connects the temporal fluctuations of the total power emitted by two incoherent sources to the CDOS, thus showing that these fluctuations encode the relative distance between the sources. This suggests that a retrieval of the amplitude of the CDOS (or equivalently of the imaginary of the Green function connecting two different points) is possible in a complex medium from a measurement of temporal fluctuations of the emitted power emerging from two incoherent sources. Such a measurement would resemble the Green function retrieval from diffuse waves that has been used in acoustics, sismology or with low-frequency electromagnetic waves [14]. In the later the statistical isotropic ambient noise is detected by two receivers, while in the method suggested here the Green function is encoded in the noise due to the fluctuations of the two sources. This results in a measurement of power fluctuations, which in the optical regime might be easier than a measurement of field-field correlations.

III. CONFIGURATIONAL AVERAGING IN A DISORDERED MEDIUM

We now assume that in addition to a temporal averaging over the fluctuations of the sources, an average over the configurations of the disordered medium can be performed. A specific situation would be that of sources embedded in a dynamic medium, with configurational changes occurring on a time scale much larger than the characteristic time of the fluctuations of the sources, as schematically shown in Fig. 2(a). An equivalent situation is that of sources moving inside a frozen disordered medium, also on a sufficiently large time scale. If both the sources and the disordered medium are fixed (i.e. the medium itself does not fluctuate), an artificial configurational averaging process could be induced by an external moving diffuser surrounding the medium, as shown in Fig. 2(b). In all these situations both averaging processes can be performed independently and subsequently. The (temporal) variance of the total emitted power averaged over the configurations of the disordered medium is readily obtained from Eq. (7):

$$\langle \overline{P^2} \rangle - \langle \bar{P}^2 \rangle = \frac{\pi^2 \omega^4}{8 \epsilon_0^2} |p|^4 \langle \rho_{12}^2 \rangle \quad (8)$$

where we use the notation $\langle \dots \rangle$ for configurational averaging.

In practice, it is often convenient to work with normalized statistical quantities. For later convenience, we can introduce the contrast of the power fluctuations, that we

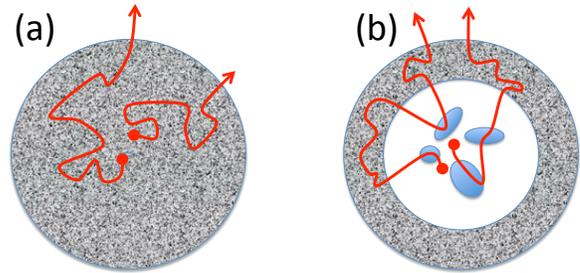


FIG. 2: Illustrations of two situations in which configurational averaging can be achieved. (a): Two sources embedded in a dynamic scattering medium, or moving inside a static disordered medium. (b): The sources are pinned in a static medium and the speckle is produced by an external diffuser surrounding the medium.

define as

$$\sigma_P = \frac{\langle \overline{P^2} \rangle - \langle \bar{P}^2 \rangle}{\langle \bar{P}^2 \rangle}. \quad (9)$$

Let us point out again the specific use of two non-commuting averaging processes in this definition, one over the temporal fluctuations of the sources and subsequently one over the configurations of the disordered medium. From Eqs. (5) and (8), the power contrast can be written in terms of the LDOS and CDOS, leading to

$$\sigma_P = \frac{2 \langle \rho_{12}^2 \rangle}{\langle (\rho_{11} + \rho_{22})^2 \rangle}. \quad (10)$$

In the next section we will derive a simple relationship between the power contrast σ_P and the speckle contrast σ_S deduced from the radiated intensity measured in a single direction, or equivalently at a single point in the far field speckle pattern.

IV. SPECKLE CONTRAST

A measurement of the total emitted power P requires a detection integrated over 4π steradians. Measuring fluctuations of the intensity radiated in a given direction, or intensity correlations between two different directions, could be a more convenient approach in practice. The time-averaged far-field intensity emerging in a given direction \mathbf{u} , per unit solid angle, and for a polarization state \mathbf{e}_α , is given by

$$\overline{I_\alpha(\mathbf{u})} = \overline{I_{1\alpha}(\mathbf{u})} + \overline{I_{2\alpha}(\mathbf{u})} \quad (11)$$

where $I_{j\alpha}(\mathbf{u})$ is the intensity radiated by the point source $\mathbf{p}_j(t)$. The latter reads as

$$I_{j\alpha}(\mathbf{u}) = \lim_{r \rightarrow \infty} \frac{\mu_0 \omega^4}{2c} r^2 |\mathbf{e}_\alpha \cdot \mathbf{G}(\mathbf{r}, \mathbf{r}_j) p_j \mathbf{u}_j|^2 \quad (12)$$

with $\mathbf{r} = r\mathbf{u}$. The configurational average of the (time averaged) total emitted power and fluctuations can be rewritten as

$$\langle \overline{P} \rangle = \int_{4\pi} \langle \overline{I(\mathbf{u})} \rangle d\mathbf{u} \quad (13)$$

$$\langle \overline{P^2} \rangle = \sum_{\alpha\alpha'} \iint_{4\pi} \langle \overline{I_\alpha(\mathbf{u})I_{\alpha'}(\mathbf{u}')} \rangle d\mathbf{u} d\mathbf{u}' \quad (14)$$

where $d\mathbf{u}$ means integration over the solid angle, α runs over the two orthogonal polarization states in the far field, and $I(\mathbf{u}) = \sum_\alpha I_\alpha(\mathbf{u})$. We now assume that the radiated field is a random statistically isotropic field, under the only constraints given by Eqs. (13) and (14). In this case one has $\langle \overline{I(\mathbf{u})} \rangle = \langle \overline{P} \rangle / (4\pi)$ and

$$\langle \overline{I_\alpha(\mathbf{u})I_{\alpha'}(\mathbf{u}')} \rangle = \frac{1}{(8\pi)^2} \langle \overline{P^2} \rangle (1 + \delta_{\alpha\alpha'}\delta_{\mathbf{u}\mathbf{u}'} \quad (15)$$

where $\delta_{\alpha\alpha'}$ is the Kronecker delta ($\sum_{\alpha\alpha'} \delta_{\alpha\alpha'} = \mathbf{e}_\alpha \cdot \mathbf{e}_{\alpha'}$) and $\delta_{\mathbf{u}\mathbf{u}'}$ is a delta function with respect to integration over the solid angle. The derivation of Eq. (15) is given in Appendix A.

At this stage, it is worth noticing that relation (15) between the speckle intensity-intensity correlation function and the fluctuations of the total emitted power is the origin of the so-called C_0 correlation known for a speckle produced by a single source [22–25]. If we consider the particular case of a single source at \mathbf{r}_1 ($\mathbf{p}_2 = 0$) that does not fluctuate in time, the normalized intensity-intensity correlation function $C(\mathbf{u}, \alpha; \mathbf{u}', \alpha')$ follows directly from Eq. (15) and is given by

$$\begin{aligned} C(\mathbf{u}, \alpha; \mathbf{u}', \alpha') &= \frac{\langle I_{1\alpha}(\mathbf{u})I_{1\alpha'}(\mathbf{u}') \rangle}{\langle I_{1\alpha}(\mathbf{u}) \rangle \langle I_{1\alpha'}(\mathbf{u}') \rangle} - 1 \\ &= C_0 + (C_0 + 1)\delta_{\alpha\alpha'}\delta_{\mathbf{u}\mathbf{u}'} \end{aligned} \quad (16)$$

where

$$C_0 = \frac{\langle \rho_{11}^2 \rangle - \langle \rho_{11} \rangle^2}{\langle \rho_{11} \rangle^2} \quad (17)$$

is the normalized variance of the LDOS at the position of the source, that is at the origin of the infinite-range C_0 contribution to the speckle correlation function [24, 25]. Therefore, for a single source, Eq. (15) provides another way of deriving the well-known results related to the C_0 correlation. Moreover, from Eq. (15), the speckle contrast (defined as the normalized variance of the intensity in a specific direction) is given by

$$\frac{\langle I(\mathbf{u})^2 \rangle - \langle I(\mathbf{u}) \rangle^2}{\langle I(\mathbf{u}) \rangle^2} = 1 + 2C_0 \quad (18)$$

a result already obtained by Shapiro [22], based on a microscopic diagrammatic approach for scalar waves. This result leads to non-universal corrections to the Rayleigh statistical distribution of the intensity in speckle patterns produced by multiple scattering [22, 26].

For two fluctuating sources, assuming again that configurational changes in the disordered medium occur on time scales larger than the characteristic time of the source fluctuations, we obtain from Eqs. (13)-(15):

$$\begin{aligned} &\langle \overline{I_\alpha(\mathbf{u})I_{\alpha'}(\mathbf{u}')} - \overline{I_\alpha(\mathbf{u})}^2 \rangle = \\ &\frac{1}{(8\pi)^2} \left\{ \langle \overline{P^2} - \overline{P}^2 \rangle + \langle \overline{P^2} \rangle \delta_{\alpha\alpha'}\delta_{\mathbf{u}\mathbf{u}'} \right\}. \end{aligned} \quad (19)$$

Making use of Eqs. (5) and (8) for two sources with the same amplitude p , we obtain

$$\begin{aligned} &\langle \overline{I_\alpha(\mathbf{u})I_{\alpha'}(\mathbf{u}')} - \overline{I_\alpha(\mathbf{u})}^2 \rangle = \\ &\frac{\omega^4}{8^3 \epsilon_0^2} |p|^4 \left\{ \langle \rho_{12}^2 \rangle + \frac{1}{2} \langle (\rho_{11} + \rho_{22})^2 + 2\rho_{12}^2 \rangle \delta_{\alpha\alpha'}\delta_{\mathbf{u}\mathbf{u}'} \right\}. \end{aligned} \quad (20)$$

This equation shows that the intensities corresponding to two different speckle spots ($\mathbf{u} \neq \mathbf{u}'$) in the angular speckle pattern formed by two incoherent point sources are strongly correlated, with a correlation given by the fluctuations $\langle \rho_{12}^2 \rangle$ of the CDOS. We end up with the surprising result that for two incoherent sources a cross-term survives averaging, and induces infinite range correlations in the speckle pattern. This correlation is formally very similar to that given by the C_0 contribution in the case of a single source, the fluctuations of the CDOS replacing the fluctuations of the LDOS. Since the CDOS is a two-point quantity, connecting in this specific situation the two source points, the speckle correlations encode the distance between the two sources.

Another implication of Eq. (20) is that in the presence of multiple scattering, CDOS fluctuations can be accessed from measurements of the directional intensity, and not only from measurements of the total power P as suggested initially by Eq. (8). More precisely, we will now show that the speckle contrast measured in a single speckle spot (hereafter denoted by σ_S) contains the same information as the contrast σ_P in Eq. (10), that assumed a measurement of the total emitted power. To proceed, we define the normalized intensity-intensity correlation function as

$$C_S(\mathbf{u}, \alpha; \mathbf{u}', \alpha') \equiv \frac{\langle \overline{I_\alpha(\mathbf{u})I_{\alpha'}(\mathbf{u}')} - \overline{I_\alpha(\mathbf{u})}^2 \rangle}{\langle \overline{I_\alpha(\mathbf{u})}^2 \rangle} \quad (21)$$

$$= \sigma_P + (\sigma_P + 1)\delta_{\alpha\alpha'}\delta_{\mathbf{u}\mathbf{u}'} \quad (22)$$

where we have made use of Eqs. (10) and (19) in the last line. The speckle contrast σ_S is readily deduced, and reads as

$$\sigma_S = \frac{\langle \overline{I(\mathbf{u})^2} \rangle - \langle \overline{I(\mathbf{u})} \rangle^2}{\langle \overline{I(\mathbf{u})} \rangle^2} = 1 + 2\sigma_P \quad (23)$$

or equivalently

$$\sigma_S = 1 + 4 \frac{\langle \rho_{12}^2 \rangle}{\langle (\rho_{11} + \rho_{22})^2 \rangle}. \quad (24)$$

This equation is the second important result in this paper. It shows that for two incoherent sources in a disordered medium, the speckle contrast measured in a single speckle spot, and computed from both temporal averaging over the fluctuations of the sources *and* configurational averaging over disorder, contains information on the CDOS that connects the two sources, and therefore on the interdistance between the sources. Since the spatial variations of the CDOS (Green function) are driven by the microscopic structure of the disordered medium, the interdistance is in principle encoded with a resolution beyond the free-space diffraction limit.

V. WEAK SCATTERING LIMIT

When the sources are embedded in a weakly disordered, homogeneous and isotropic medium, it is possible to give an explicit approximated expression of the fluctuations of the CDOS $\langle \rho_{12}^2 \rangle = 4\omega^2 / (\pi^2 c^4) \langle (\text{Im}G_{12})^2 \rangle$. In the limit $k_0 n_{\text{eff}} \ell \gg 1$, where n_{eff} is the effective refractive index, $k_0 = 2\pi/\lambda$ (with λ the wavelength in vacuum) and ℓ the Boltzmann scattering mean free path, the averaged Green function is given by the well known result for homogeneous and isotropic media and we can write $\langle \text{Im}G_{jk} \rangle$ as

$$\langle \text{Im}G_{jk} \rangle \approx \left(\mathbf{u}_1 \cdot \mathbf{u}_2 + \frac{(\mathbf{u}_1 \cdot \nabla_{\mathbf{r}_1})(\mathbf{u}_2 \cdot \nabla_{\mathbf{r}_2})}{k_{\text{eff}}^2} \right) \frac{\sin(k_{\text{eff}} r_{12})}{4\pi r_{12}} \quad (25)$$

where $r_{12} = |\mathbf{r}_2 - \mathbf{r}_1|$ and $k_{\text{eff}} = k_0 n_{\text{eff}} + i/(2\ell)$ is the effective wavenumber. This approximation breaks down when the distance between dipoles, r_{12} , approach their effective size (see the discussion in Ref. [27] and references therein). At lowest order, we can further approximate $\langle \text{Im}G_{jk}^2 \rangle \approx \langle \text{Im}G_{jk} \rangle^2$ which, from Eqs. (20), or (24), and (25) provides an explicit expression of the intensity correlation function, or the speckle contrast, in terms of the relative distance between the sources.

VI. CONCLUSION

The results derived in this paper suggest the intriguing possibility that intensity measurements at only one point in a speckle pattern produced by two incoherent sources can provide information about the relative distance between the sources. Moreover, this information is in principle not limited by diffraction. It can be extracted from the speckle contrast σ_S obtained after a proper time and configurational averaging process. The results also suggests an alternative approach to the Green function retrieval technique. In the later the statistical isotropic ambient noise is detected by two receivers, while in the method suggested here the Green function is encoded in the noise due to two fluctuating incoherent sources measured at a single point. Finally, let us note that for

fluorescent emitters, the quantum optical equivalent of Eqs. (7), (10) or (24) might be established in terms of decay rate or photocount statistics of a coupled system, and fluctuations could be associated with super-radiant and sub-radiant states. This quantum treatment is left for future work.

Acknowledgments

We acknowledge helpful discussions with M. Fink and F. Scheffold. This work was supported by the Spanish MEC through Grant No. FIS2012-36113, by the Comunidad de Madrid through Grant No. S2009/TIC-1476 (Microseres Project), by LABEX WIFI (Laboratory of Excellence within the French Program "Investments for the Future") under references ANR-10-LABX-24 and ANR-10-IDEX-0001-02 PSL*. JJS acknowledges an IKERBASQUE Visiting Fellowship and RC and GC the hospitality of the DIPC at Donostia-San Sebastian (Spain) where part of this work was done.

Appendix A: Speckle intensity correlations: Derivation of Eq. (15) from Random Matrix Theory

The far-field intensity radiated by two dipoles emerging in a given direction \mathbf{u} , per unit solid angle, and for a polarization state \mathbf{e}_α , can be written as

$$I_\alpha(\mathbf{u}) = \lim_{r \rightarrow \infty} \frac{\mu_0 \omega^4}{2c} r^2 \left| \sum_{j=1}^2 \mathbf{e}_\alpha \cdot \mathbf{G}(\mathbf{r}, \mathbf{r}_j) \mathbf{p}_j \right|^2 \quad (A1)$$

with $\mathbf{r} = r\mathbf{u}$. Assuming that the 4π solid angle is divided in a finite number $N \gg 1$, of (solid angle) pixels, the total power emitted by the two sources is given by

$$P = \int_{4\pi} I(\mathbf{u}) d\mathbf{u} = \frac{4\pi}{N} \sum_{\alpha} \sum_u I_\alpha(\mathbf{u}) \quad (A2)$$

which can be written in compact matrix form,

$$P = \frac{4\pi}{N} \hat{\mathbf{p}}^\dagger \Psi^\dagger \Psi \hat{\mathbf{p}} \quad (A3)$$

where $\hat{\mathbf{p}}^\dagger = (\mathbf{p}_1^\dagger, \mathbf{p}_2^\dagger) = (p_{1x}^*, p_{1y}^*, p_{1z}^*, p_{2x}^*, p_{2y}^*, p_{2z}^*)$ and Ψ is a $2N \times 6$ matrix ($2N$ comes from the two orthogonal polarizations of the field for each observation angle in the far-field). By using the singular value decomposition (SVD), the Ψ matrix can be factorized as

$$\Psi = \mathbf{U} \Sigma \mathbf{V}^\dagger \quad (A4)$$

$$\Psi^\dagger \Psi = \mathbf{V} \Sigma^2 \mathbf{V}^\dagger \quad (A5)$$

where \mathbf{U} is an unitary $2N \times 2N$ matrix, Σ is a $2N \times 6$ rectangular diagonal matrix with nonnegative real numbers on the diagonal and \mathbf{V} is a 6×6 unitary matrix.

Equation (4) in the main text can now be written as

$$P = \hat{\mathbf{p}}^\dagger \boldsymbol{\Psi}^\dagger \boldsymbol{\Psi} \hat{\mathbf{p}} = \hat{\mathbf{p}}^\dagger \mathbf{V} \boldsymbol{\Sigma}^2 \mathbf{V}^\dagger \hat{\mathbf{p}} = \frac{\mu_0 \omega^3}{2} \sum_{j,j'=1}^2 p_j^* p_j \text{Im} G_{jj'} \\ = \frac{\mu_0 \omega^3}{8\pi} \begin{pmatrix} \mathbf{p}_1^\dagger & \mathbf{p}_2^\dagger \end{pmatrix} \begin{bmatrix} \text{Im} \mathbf{G}(11) & \text{Im} \mathbf{G}(12) \\ \text{Im} \mathbf{G}(21) & \text{Im} \mathbf{G}(22) \end{bmatrix} \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{pmatrix} \quad (\text{A6})$$

where $\mathbf{G}(jj') \equiv \mathbf{G}(\mathbf{r}_j, \mathbf{r}_{j'})$. In terms of these new matrices, the intensity at a given angle \mathbf{u} and polarization α can be computed as

$$I_\alpha(\mathbf{u}) = \sum_{a,b,a',b'=1}^6 \hat{p}_{a'}^\dagger V_{a'b'} \Sigma_{b'b'} U_{b'u_\alpha}^\dagger U_{u_\alpha b} \Sigma_{bb} V_{ba}^\dagger \hat{p}_a \quad (\text{A7})$$

where \hat{p}_a corresponds to the i_a component of the dipole \mathbf{p}_{j_a} (with $j_a = \text{int}[(a+2)/3]$ and $i_a = a + 3 - 3j_a$).

If we assume that the radiated field is statistically isotropic, we can consider \mathbf{U} as a random unitary matrix statistically independent of the $\boldsymbol{\Sigma}$ and \mathbf{V} matrices (isotropy hypothesis). This is one of the key assumptions in the macroscopic scaling approach of transport theory [30–32]. By using the averages over the unitary group (evaluated by Mello in ref. [31]),

$$\langle U_{b'u_\alpha}^\dagger U_{u'_\alpha b} \rangle = \frac{1}{2N} \delta_{bb'} \delta_{\mathbf{u}\mathbf{u}'} \delta_{\alpha\alpha'} \quad (\text{A8})$$

and the ensemble average of the intensity is given by

$$\langle I_\alpha(\mathbf{u}) \rangle = \frac{1}{2N} \sum_{a,b,a'=1}^6 \hat{p}_{a'}^\dagger \langle V_{a'b} \Sigma_{bb}^2 V_{ba}^\dagger \rangle \hat{p}_a \quad (\text{A9})$$

which, from Eq. (A6), gives

$$\langle I_\alpha(\mathbf{u}) \rangle = \frac{\mu_0 \omega^3}{16\pi} \hat{\mathbf{p}}^\dagger [\langle \text{Im} \mathbf{G} \rangle] \hat{\mathbf{p}} = \frac{1}{2} \frac{\langle P \rangle}{4\pi} \quad (\text{A10})$$

with $\hat{\mathbf{p}} = \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{pmatrix}$ and

$$\langle \text{Im} \mathbf{G} \rangle = \begin{pmatrix} \langle \text{Im} \mathbf{G}(11) \rangle & \langle \text{Im} \mathbf{G}(12) \rangle \\ \langle \text{Im} \mathbf{G}(21) \rangle & \langle \text{Im} \mathbf{G}(22) \rangle \end{pmatrix} \quad (\text{A11})$$

i.e., as expected, under the isotropy hypothesis, the intensity at a given angle is simply proportional to the average of the total radiated power, $\langle I(\mathbf{u}) \rangle = 2 \langle I_\alpha(\mathbf{u}) \rangle = \langle P \rangle / (4\pi)$. After averaging Eq. (A7) over the temporal fluctuations of the sources we have $\overline{\hat{p}_a \hat{p}_b} = \delta_{j_a j_b}$ and $\overline{I_\alpha(\mathbf{u})}$ is simply the sum of the individual intensities [Eq. (11)] i.e. the non-diagonal boxes of $\langle \text{Im} \mathbf{G} \rangle$ do not contribute (no crosstalk term). The same applies for

$$\overline{\langle I_\alpha(\mathbf{u}) \rangle} = \overline{\langle I_\alpha(\mathbf{u}) \rangle} = \overline{\langle P \rangle} / (8\pi) = \overline{\langle P \rangle} / (8\pi) \quad (\text{A12}) \\ = \frac{\mu_0 \omega^3}{16\pi} \begin{pmatrix} \mathbf{p}_1^\dagger & \mathbf{p}_2^\dagger \end{pmatrix} \begin{pmatrix} \langle \text{Im} \mathbf{G}(11) \rangle & \mathbf{0} \\ \mathbf{0} & \langle \text{Im} \mathbf{G}(22) \rangle \end{pmatrix} \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{pmatrix} \quad (\text{A13})$$

Let us now consider Eq. (14),

$$\overline{\langle P^2 \rangle} = \frac{(4\pi)^2}{N^2} \sum_{\alpha\alpha'} \sum_{\mathbf{u}, \mathbf{u}'} \overline{\langle I_\alpha(\mathbf{u}) I_{\alpha'}(\mathbf{u}') \rangle}, \quad (\text{A14})$$

with $I_\alpha(\mathbf{u})$ given by Eq. (A7). Computation of Eq. (A14) involves again $\overline{\hat{p}_a \hat{p}_b} = \delta_{j_a j_b}$ together with averages of 4 elements of an unitary random matrix [31]

$$\langle U_{b'u_\alpha}^\dagger U_{u_\alpha b} U_{c'u'_\alpha}^\dagger U_{u'_\alpha c} \rangle = \frac{\delta_{bb'} \delta_{cc'} + \delta_{bc'} \delta_{cb'} \delta_{u_\alpha u'_\alpha}}{4N^2 - 1} \\ - \frac{\delta_{bc'} \delta_{cb'} + \delta_{bb'} \delta_{cc'} \delta_{u_\alpha u'_\alpha}}{2N(4N^2 - 1)}. \quad (\text{A15})$$

It is easy now to find:

$$\overline{\langle I_\alpha(\mathbf{u}) I_{\alpha'}(\mathbf{u}') \rangle} = \frac{N}{2(2N+1)} \frac{1}{(4\pi)^2} \overline{\langle P^2 \rangle} \left(1 + \delta_{\mathbf{u}\mathbf{u}'} \delta_{\alpha\alpha'} \right) \quad (\text{A16})$$

which, in the large N limit, gives Equation (15) in the main text.

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