Multifractality of wavefunctions
at the Anderson transition

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The “kicked rotor” collaboration (Paris-Lille)

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Outline

- Anderson localization with cold atoms
- The periodically kicked rotor: dynamical localization and Anderson localization
- Anderson localization and Anderson transition for the “three-color” kicked rotor
- Critical regime of the Anderson transition
- Multifractality and fluctuations of the critical wavefunctions
Anderson (a.k.a. Strong) localization

- Particle in a disordered (random) potential:
  - One-dimensional system
    - Particle with energy $E$
  - Two-dimensional system
  - Disordered potential $V(z)$ (typical value $V_0$)

- When $E \ll V_0$, the particle is classically trapped in the potential wells.
- When $E \gg V_0$, the classical motion is ballistic in 1d, typically diffusive in dimension 2 and higher.
- Quantum interference may inhibit diffusion at long times => Anderson localization
Anderson localization in 1d

- Anderson localization is the **generic** behaviour, even for very small disorder!
- Simple Anderson model: discretized Schrödinger equation on a 1d lattice.

\[
H = \sum_n \epsilon_n |n\rangle \langle n| + t |n\rangle \langle n+1| + t |n+1\rangle \langle n|
\]

\( t=1: \) hopping

\[ \bullet \ldots \bullet \] Sites

\( \epsilon_n \): diagonal disorder
Anderson localization in 1d

- Propagation of an initially localized wave-packet:

\[ |\psi(z, t = 10000)|^2 \]

\[ |\psi(z, t = 20000)|^2 \]

- Initial Gaussian wavepacket

- Log scale

- Exponential localization at long time

- \[ \langle z^2 \rangle(t) \] and \[ |\psi(z, t)|^2 \] display signatures of Anderson localization
Anderson localization in 1d

- When averaged over time and/or different realizations of the disorder, the fluctuations are smoothed out:

$$|\psi(z)|^2$$

Averaged over 800 realizations of the disorder

Log scale
Anderson localization in 1d and 2d

- Anderson localization is due to interference between various multiple scattering paths:
  - requires perfect phase coherence.
  - not restricted to quantum matter waves. It has been observed with several types of waves: microwaves, light, acoustic waves, seismic waves, etc.
  - The importance of interference terms depends crucially on the geometry of multiple scattering paths, i.e. on the dimension.
- A detailed understanding can be obtained from the scaling theory of localization (Abrahams, Anderson, Licciardello, Ramakrishnan, 1979).

- **Dimension 1**: (almost) always localized.
  - Localization length \( \xi_{\text{loc}} = 2\ell \)
  - \( \ell \): mean free path.
- **Dimension 2**: marginally localized.
  - Localization length \( \xi_{\text{loc}} \approx \ell \exp \left( \frac{\pi k \ell}{2} \right) \)
Anderson localization in 3d (and beyond)

- For **weak disorder**, the quantum motion is diffusive (weak localization) => metallic behaviour.
- For **strong disorder**, Anderson localization takes place => insulator.
- The metal-insulator transition (Anderson transition) takes place at the “mobility edge”. It is controlled by the $k\ell$ parameter. The critical point is approximately given by:
  \[(k\ell)_c = 1 \text{ Ioffe-Regel criterion}\]
- The Anderson transition is a second-order transition. On the insulating side, the localization length diverges like:
  \[\xi_{loc} \sim \frac{1}{[(k\ell)_c - k\ell]^{\nu}} \quad \nu : \text{critical exponent}\]
- On the metallic side: \[D \sim [k\ell - (k\ell)_c]^s \quad \text{with} \quad s = (d - 2)\nu\]
- Numerical results suggest \(\nu \approx 1.60\) .
Anderson localization with cold atoms

Essential features of Anderson localization:

- Inhibition of transport
  - direct measurement of the atomic wavefunction
- Due to quantum interference
  - preserve phase coherence of the atomic wavefunction
- Zero-temperature effect
  - cold atomic gas
- One-body physics
  - dilute gas (no Bose-Einstein condensate)
- Driven by the amount of disorder
  - create a tunable disorder using light-atom interaction
Quantum dynamics of the external motion of cold atoms

- Control of the dynamics with laser fields, magnetic fields, gravitational field.
- Orders of magnitude:
  - Velocity: cm/s
  - De Broglie wavelength: µm
  - Time: µs-ms
- One-body (if sufficiently dilute, avoid BEC) zero-temperature quantum dynamics with small decoherence.
- The light-shift due to a quasi-resonant laser at frequency $\omega_L = \omega_{at} - \delta$ is proportional to $I/\delta$, (with $I$ the space-dependent laser intensity) and acts as an effective potential.
- Inelastic processes scale as $I/\delta^2$, thus can be made negligible if a large detuning (and laser power!) is used.
Anderson localization with cold atoms

Essential features of Anderson localization:

- Inhibition of transport
  - direct measurement of the atomic wavefunction
- Due to quantum interference
  - preserve phase coherence of the atomic wavefunction
- Zero-temperature effect
  - cold atomic gas
- One-body physics
  - dilute gas (no Bose-Einstein condensate)
- Driven by the amount of disorder
  - use chaotic temporal dynamics instead of static disorder
- Depends on the dimension
  - effective dimension can be easily adjusted by varying the temporal sequence
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The (periodically) kicked rotor

- Hamiltonian: $$H = \frac{p^2}{2} + k \cos \theta$$
- Map from kick \( n \) to kick \( n+1 \):
  \[
  \begin{align*}
  Tp_{n+1} &= Tp_n + kT \sin \theta_n \\
  \theta_{n+1} &= \theta_n + Tp_{n+1}
  \end{align*}
  \]

  **Standard map**
  **stochasticity parameter**
  $$K = kT$$

  - The classical dynamics is regular for low \( K \), and almost fully chaotic for \( K > 4 \).
  - At large \( K \), it is analogous to a random walk in momentum space => **chaotic diffusion**
    \[
    \langle p_n^2 \rangle - \langle p_0^2 \rangle \approx nk^2 / 2
    \]

The kicked rotor is periodic in \( \theta \), the spatially unfolded version is not => requires the inclusion of a (constant) quasi-momentum in the quantum treatment (Bloch theorem).
Quantum dynamics of the kicked rotor

- Numerical experiment: compare classical and quantum dynamics (averaged over an ensemble of initial conditions).
- Start from a state well localized in momentum space.

\[ |\psi(p)|^2 \] (log scale)

\[ \langle p^2(t) \rangle \]
Quantum dynamics of the kicked rotor

- Numerical experiment: compare classical and quantum dynamics (averaged over an ensemble of initial conditions).
- Start from a state well localized in momentum space.
- At short time, the quantum and classical dynamics are equivalent.

\[ |\psi(p)|^2 \]  
(log scale)

Classical dynamics
Quantum dynamics

\[ \langle p^2(t) \rangle \]  
Time \( t \) (number of kicks)
Quantum dynamics of the kicked rotor

- Numerical experiment: compare classical and quantum dynamics (averaged over an ensemble of initial conditions).
- Start from a state well localized in momentum space.
- At long time, the quantum dynamics freeze.
- Equivalent to Anderson localization in momentum space (Fishman et al, 1982)

\[ |\psi(p)|^2 \]
(log scale)

Quantum dynamics: dynamical localization

Casati et al. (1979)

Classical dynamics: chaotic diffusion

Time \( t \) (number of kicks)
Dynamical vs. Anderson localization

- The Hamiltonian is time-periodic => use Floquet theorem.
- Floquet states: eigenstates of the one-period evolution operator:

\[ U |\phi_i\rangle = \exp \left( -\frac{iE_i T}{\hbar} \right) |\phi_i\rangle \quad \frac{2\pi\hbar}{T} > E_i \geq 0 \]

- In the momentum eigenbasis (labelled by integer \( m \), wavefunction components \( \chi_m \)), the eigen-equation becomes:

\[
\epsilon_m \chi_m + \sum_{r \neq 0} W_r \chi_{m-r} = -W_0 \chi_m
\]

where \( W(\theta) = \tan \left( \frac{k \cos \theta}{2\hbar} \right) = \sum_{r=-\infty}^{\infty} W_r \exp (ir\theta) \)
represents the coupling between various momentum states.

- \( \epsilon_m = \tan \left[ \frac{\left(E - \frac{m^2\hbar^2}{2}\right) T}{2\hbar} \right] \) is the on-site energy.
Dynamical vs. Anderson localization (II)

\[ \epsilon_m \chi_m + \sum_{r \neq 0} W_r \chi_{m-r} = -W_0 \chi_m \]

- The \( W_r \) decrease rapidly at large \( r \) and the \( \epsilon_m \) are pseudo-random variables \( \Rightarrow \) similar to Anderson model.

- Various Floquet states correspond to various realizations of the disorder, but have the same localization length.

- The hopping integrals \( W_r \) increase with the kick strength \( K \) \( \Rightarrow \) \( K \) plays the role of a control parameter of the Anderson model.

- More an analogy than a real proof.
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The kicked rotor using cold atoms

- Create a spatially periodic optical potential using a standing laser wave.
- Temporal modulation of the laser intensity (with $\delta$-kicks) $\Rightarrow$ kicked rotor with adjustable effective Plank constant $\tilde{\hbar}_{\text{eff}} = 8\omega_r T$ ($\omega_r$: atomic recoil frequency, $T$: kicking period).
- Experimental setup:
  
  1. Prepare a cold Cs cloud in a Magneto-Optical Trap;
  2. Switch off MOT and magnetic field;
  3. Apply the sequence of kicks (10-200 kicks);
  4. Analyze momentum distribution along the laser axis (using velocity selective Raman transitions);

- Atoms fall down because of gravitational field $\Rightarrow$ 200 kicks maximum.
- Sources of decoherence: spontaneous emission, residual gravitational field along the laser axis are small enough over 200 kicks.
How to observe the Anderson transition in 3D?

- Dynamical vs. Anderson localization
  - Anderson localization: 1d disordered static $x$-space
  - Dynamical localization: 1d chaotic time-periodic $p$-space
- Simple idea: keep the spatial dynamics one-dimensional, but introduce one or several additional temporal dimensions.

\[
H = \frac{p^2}{2} + k \cos \theta \left(1 + \epsilon \cos \omega_2 t \cos \omega_3 t\right) \sum_n \delta(t - nT)
\]

- Substantially equivalent with the Anderson model in higher dimension (Casati, Guarneri and Shepelyansky, PRL, 1989).
- Prediction of a localized-delocalized Anderson transition when $K = kT$ is increased in a 3-color system.
Mapping of the quasiperiodic kicked rotor on the 3D Anderson model

- The temporal evolution of the state $\psi(\theta)$ of the 1D quasiperiodic kicked rotor with Hamiltonian:
  \[
  H = \frac{p^2}{2} + K \cos \theta (1 + \epsilon \cos \omega_2 t \cos \omega_3 t) \sum_n \delta(t - n)
  \]
is identical to the one of the state:
  \[
  \Psi(x_1, x_2, x_3) = \psi(x_1 = \theta) \delta(x_2) \delta(x_3)
  \]
of the 3D periodic kicked "rotor":

\[
\mathcal{H} = \frac{p_1^2}{2} + \omega_2 p_2 + \omega_3 p_3 + K \cos x_1 [1 + \epsilon \cos x_2 \cos x_3] \sum_n \delta(t - n)
\]

**Pseudo 3D "rotor"**

- The initial state is completely delocalized in the transverse $p_2$ and $p_3$ directions.
- The classical dynamics of the 3D periodic kicked "rotor" is an anisotropic chaotic diffusion in momentum space.
Mapping of the quasiperiodic kicked rotor on the 3D Anderson model

- Similarly to the 1D periodic kicked rotor, the Schroedinger equation for the Floquet states of the 3D periodic kicked “rotor” can be written as:

$$\epsilon_m \Phi_m + \sum_{r \neq 0} W_r \Phi_{m-r} = -W_0 \Phi_m \quad \text{for} \quad m = (m_1, m_2, m_3)$$

with diagonal “disorder”:

$$\epsilon_m = \tan \left\{ \frac{1}{2} \left[ \omega - \left( \frac{\hbar m_1^2}{2} + \omega_2 m_2 + \omega_3 m_3 \right) \right] \right\}$$

and the hopping coefficients $W_m$ are the Fourier components of:

$$W(x_1, x_2, x_3) = \tan \left[ K \cos x_1 (1 + \epsilon \cos x_2 \cos x_3) / 2\hbar \right]$$

- Effective 3D Anderson model, with a metal-insulator transition. For details, see Lemarié et al, PRA, 80, 043626 (2009), arXiv:0907.3411.
Schematic view of the experiment

Kicks (amplitude quasi-periodically modulated with time)

\[ H = \frac{p^2}{2} + k \cos \theta \left( 1 + \epsilon \cos \omega_2 t \cos \omega_3 t \right) \sum_n \delta(t - nT) \]
Numerical results for the three-color kicked rotor

Momentum distribution

$t=105$

$K$ (kick strength)
Numerical results for the three-color kicked rotor

Momentum distribution

Localized regime

$K$ (kick strength)

Diffusive regime
How to identify unambiguously the transition?

\[ |\psi(p)|^2 \]

\[ \langle p^2(t) \rangle \]

3 increasing \( K \) values

Time (number of kicks)
How to identify unambiguously the transition?

\[ |\psi(p)|^2 \]

\[ \langle p^2(t) \rangle \]

Time (number of kicks)

- At criticality, one expects an anomalous diffusion with (see below)

\[ \langle p^2(t) \rangle \sim t^\gamma \quad \text{with} \quad \gamma = \frac{2}{3} \]
From localization to diffusive regime: experimental results

\[ \frac{1}{\Pi_0^2(t)} \propto \langle p^2(t) \rangle \]

- **Diffusive regime**: \( \langle p^2(t) \rangle \sim t \)
- **Critical regime**: \( \langle p^2(t) \rangle \sim t^{2/3} \)
- **Localized**
From localized to diffusive regime

Experimental results

Numerical results

\[ \frac{1}{\Pi_0^2(t)} \propto \langle p^2(t) \rangle \]

log scale

time (number of kicks, log scale)

localized regime (slope 0)
diffusive regime (slope 1)
critical regime (slope 2/3)
localized regime (slope 0)
Universality of the Anderson transition

- The Anderson transition is universal, i.e. its characteristic properties – such as the critical exponent - do not depend on the microscopic details of the system.
- We have numerically checked this universality for the quasi-periodically kicked rotor, by varying the effective Planck's constant, irrational ratio between the three frequencies, and the depth $\varepsilon$ of the modulation.
- We always observe (G. Lemarié et al, Europhys. Lett. 87, 37007 (2009), arXiv:0904.2324)

$$\nu = 1.58 \pm 0.02$$

- It is identical to the exponent numerically measured on the Anderson model (Slevin and Ohtsuki, 1999).
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Momentum distribution at the critical point

- The initial state is almost perfectly localized in momentum space => the momentum distribution at time $t$ is nothing but a direct measure of the average intensity Green function $G(0, p; t)$
- Numerical experiment at the critical point:

- Time invariant shape (neither Gaussian, nor exponential)
Momentum distributions at criticality

Distributions at various times

Distributions at various times rescaled by the critical $t^{1/3}$ law
Experimental measurements in the critical regime

- Characterized by a specific scaling: $p \propto t^{1/3}$

Critical wavefunction

- At the critical point, the diffusion constant scales like:
  \[ D(\omega) \propto (-i\omega)^{1/3} \]

- Self-consistent theory of localization (à la Vollhardt-Wölfle) predicts at the critical point of the Anderson metal-insulator transition:

  \[
  |\psi(p,t)|^2 = \frac{3}{2} \left( 3\rho^{3/2} t \right)^{-1/3} \text{Ai} \left[ \left( 3\rho^{3/2} t \right)^{-1/3} |p| \right]
  \]

- Behaves asymptotically like \( \exp(-\alpha |x|^{3/2}) \)

- In excellent agreement with numerics and experimental results
Experimentally measured critical Green function

Analytical prediction (Airy function)

Experimental points (with error bars)

Residual /Airy function
Residual /Gaussian
Residual /Exponential

Rescaled momentum $p/t^{1/3}$

Lemarié et al, PRL 105, 090601 (2010)
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Momentum distribution at long time (critical regime)

Extra probability near \( p=0! \)

Airy function

The population at zero velocity slightly deviates from the \( t^{1/3} \) prediction of the scaling theory of localization
Extra peak near zero-momentum at long time

The “extra” peak is very narrow and grows with time.
Extra peak near zero-momentum at long time

The “extra” central peak grows and becomes narrower at long time.
The extra peak essentially vanishes off criticality, even for relatively short time.
At long time in the diffusive regime, no extra peak on top of the Gaussian distribution => extra peak exists only in the very vicinity of the critical point.
Possible experimental observation of the “extra” peak

Momentum distribution

Scaled momentum \( p = t^{1/3} \)

Small extra peak at short time

\( \Rightarrow \) experimental observation will require excellent signal/noise
Beyond the average Green function: multifractality of the wavefunctions

- At the critical point, the eigenstates display strong **fluctuations**:
  - Regions where the wavefunction is exceptionally **large**;
  - Regions where the wavefunction is exceptionally **small**;
  - Typical **multifractal** behaviour.
- Can be quantitatively studied using the multifractality spectrum.
- Finite system of size $L$ in dimension $d$.
  - In the diffusive (metallic) regime, the average behaviour is:
    \[
    |\psi(\mathbf{r})|^2 = \frac{1}{L^d}
    \]
  - In the localized (insulating) regime:
    \[
    |\psi(\mathbf{r})|^2 \propto \frac{1}{\xi_{\text{loc}}^d} \exp \left( -\frac{||\mathbf{r} - \mathbf{r}_0||}{\xi_{\text{loc}}} \right)
    \]
- Multifractality spectrum $f(\alpha)$: measure of the regions of space where:
  \[
  |\psi(\mathbf{r})|^2 \propto L^{-\alpha}
  \]
At the critical point, the whole curve scales with the system size.
Multifractality spectrum for the 3D Anderson model at the critical point

Fixed system size $L$

Rescaled data for various sizes

Eigenstates of the critical Anderson model

Rodriguez et al, PRL 102, 106406 (2009)
Multifractality spectrum for the quasi-periodic kicked rotor

- No direct access to the eigenstates => study the dynamics of wavepackets as a function of time
- Time plays the role of the system size.
- At long time, the “incoherent” superposition of Floquet states reduces the fluctuations of the wavepacket.

\[ |\psi(t)\rangle = \sum_n \exp \left( -i \frac{E_n t}{\hbar} \right) |\phi_n(t)\rangle \]

Wavepacket \quad Floquet eigenstates \quad |\phi_n(t + T)\rangle = |\phi_n(t)\rangle

- Additional complication: the mapping of the 3D Anderson model on the 1D quasi-periodic kicked rotor requires a 3D->1D projection (averaging) => reduced fluctuations.
Mapping of the quasiperiodic kicked rotor on the 3D Anderson model

- Hamiltonian of the 1D quasiperiodic kicked rotor:

\[ H = \frac{p_1^2}{2} + K \cos x_1 \left( 1 + \epsilon \cos \omega_2 t \ \cos \omega_3 t \right) \sum_n \delta(t - n) \]

- Hamiltonian of the 3D periodic kicked “rotor”:

\[ \mathcal{H} = \frac{p_1^2}{2} + \omega_2 p_2 + \omega_3 p_3 + K \cos x_1 \left[ 1 + \epsilon \cos x_2 \cos x_3 \right] \sum_n \delta(t - n) \]

- Temporal evolution:

\[
\begin{align*}
\text{Quasiperiodic 1D rotor} & \quad \text{Periodic 3D pseudo-rotor} \\
\psi(x_1, t = 0) & \quad \psi(x_1, t = 0) \delta(x_2) \delta(x_3) \\
\downarrow & \quad \downarrow \\
\psi(x_1, t) & \quad \psi(x_1, t) \delta(x_2 - \omega_2 t) \delta(x_3 - \omega_3 t)
\end{align*}
\]

The Floquet states of the periodic 3D “rotor” are delocalized in all directions \((x_1, x_2, x_3)\) => how are multifractal fluctuations affected when delocalized Floquet states are recombined in a wavepacket localized in \(x_2\) and \(x_3\)?
Multifractality spectrum for the quasiperiodic kicked rotor (numerics)

- Metallic (diffusive) regime: no multifractality, the multifractal spectrum tends to a single peak at $\alpha = 1$

- Critical regime at the Anderson transition: some small deviations appear, multifractal effects are nevertheless small.
  - Careful numerics needed
  - Requires to understand the effects of “folding” and “mixture of eigenstates”

P. Akridas et al, preliminary results
(see also I. Garcia-Mata et al, arXiv:1007.1404)
Summary

- Cold atoms are good systems for studying quantum interference, quantum chaos and the quantum dynamics of complex systems.
- Experimental observation of a transition from the localized to the diffusive regime, in a one-body system at zero-temperature, due to quantum interference, driven by the amount of disorder in the system => Anderson transition
- No characteristic scale at the critical point. Self-consistent theory of localization accurately predicts the average intensity Green function at short time. Excellent agreement with experimental observations.
- Small deviations from the scaling laws are numerically observed at the critical point. They are probably related to strong multifractal fluctuations at criticality:
  - Anomalous scaling of the zero-velocity population
  - “Extra” narrow peak appearing at very long time near zero velocity.
  - Multifractality spectrum of “wavepackets” at criticality.
Perspectives

- Fluctuations of the wave function at the critical point (multifractality). Can be measured in principle, but...
- What about two dimensions? Experiment in progress.
- Adding additional temporal quasi-periods => Anderson model in dimension 4, 5...
  - Numerics in $d=4 \quad \nu = 1.15 \pm 0.03$
    - Good agreement with Anderson model
  - Most probably, the upper critical dimension is infinite.
- Controlled addition of decoherence.
- Interaction between atoms (dynamical localization of a Bose-Einstein condensate) => possible modification of the critical exponent.
- Effect of quantum statistics (bosons vs. fermions).
The quantum kicked rotor

- Hamiltonian \[ H = \frac{p^2}{2} + k \cos \theta \sum_{n=-\infty}^{+\infty} \delta(t - nT) \]

- Evolution operator over one period:

\[ U = \exp \left( \frac{-ip^2T}{2\hbar} \right) \times \exp \left( \frac{-ik \cos \theta}{\hbar} \right) \]

- Very easy to numerically generate the quantum evolution.
Experimental setup

Standard Magneto-Optical Trap