Role of scattering in correlation-based imaging

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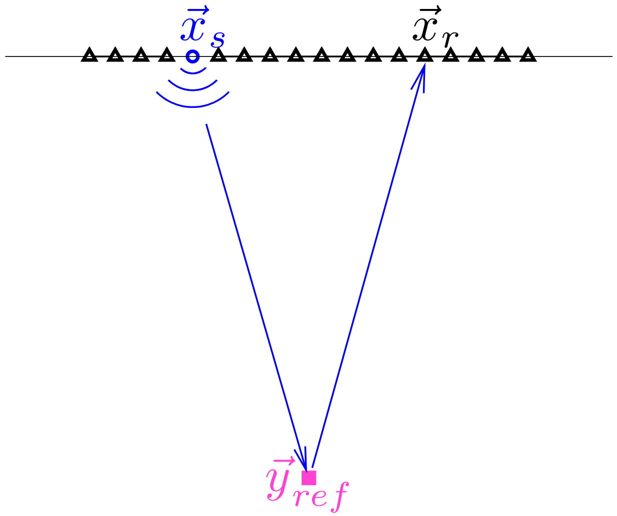
Imaging through a homogeneous medium



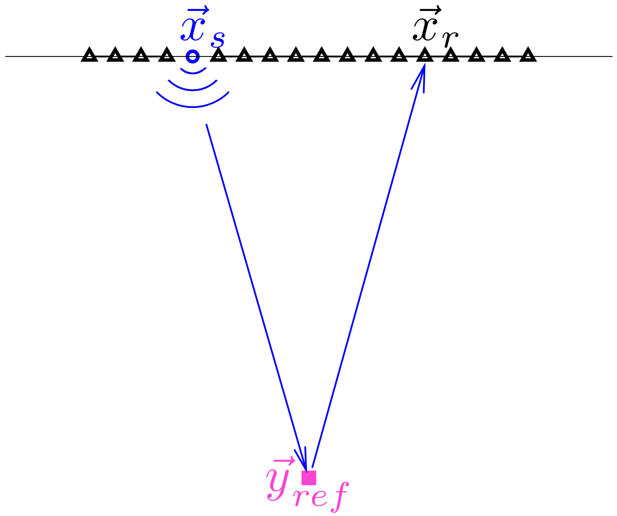
- Sensor array imaging of a reflector located at $\vec{\boldsymbol{y}}_{\rm ref}$. $\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_r$ is a receiver.

Measured data: $\{u(t, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s), r = 1, \dots, N_{\rm r}, s = 1, \dots, N_{\rm s}\}$.

- Mathematical model:

$$\left(\frac{1}{c_0^2} + \frac{1}{c_{\rm ref}^2}\mathbf{1}_{B_{\rm ref}}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{y}}_{\rm ref})\right)\frac{\partial^2 u}{\partial t^2}(t, \vec{\boldsymbol{x}}; \vec{\boldsymbol{x}}_{\rm s}) - \Delta_{\vec{\boldsymbol{x}}} u(t, \vec{\boldsymbol{x}}; \vec{\boldsymbol{x}}_{\rm s}) = f(t)\delta(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_{\rm s})$$

Imaging through a homogeneous medium



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$$\Big(\frac{1}{c_0^2}+\frac{1}{c_{\rm ref}^2}\mathbf{1}_{B_{\rm ref}}(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}}_{\rm ref})\Big)\frac{\partial^2 u}{\partial t^2}(t,\vec{\boldsymbol{x}};\vec{\boldsymbol{x}}_{\rm s})-\Delta_{\vec{\boldsymbol{x}}}u(t,\vec{\boldsymbol{x}};\vec{\boldsymbol{x}}_{\rm s})=f(t)\delta(\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_{\rm s})$$

- Image with Kirchhoff Migration:

$$\mathcal{I}_{\rm KM}(\vec{\boldsymbol{y}}^S)=\sum_{r=1}^{N_{\rm r}}\sum_{s=1}^{N_{\rm s}}u\big(\mathcal{T}(\vec{\boldsymbol{x}}_s,\vec{\boldsymbol{y}}^S)+\mathcal{T}(\vec{\boldsymbol{y}}^S,\vec{\boldsymbol{x}}_r),\vec{\boldsymbol{x}}_r;\vec{\boldsymbol{x}}_s\big)$$

It forms the image with the superposition of the backpropagated traces.
$\mathcal{T}(\vec{\boldsymbol{y}}^S,\vec{\boldsymbol{x}})$ is the travel time from $\vec{\boldsymbol{x}}$ to $\vec{\boldsymbol{y}}^S$, i.e. $\mathcal{T}(\vec{\boldsymbol{y}}^S,\vec{\boldsymbol{x}})=|\vec{\boldsymbol{y}}^S-\vec{\boldsymbol{x}}|/c_0$.

Kirchhoff Migration:

$$\mathcal{I}_{\rm KM}(\vec{\boldsymbol{y}}^S) = \sum_{r=1}^{N_{\rm r}} \sum_{s=1}^{N_{\rm s}} u\big(\mathcal{T}(\vec{\boldsymbol{x}}_s, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{y}}^S, \vec{\boldsymbol{x}}_r), \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s\big)$$

• Resolution analysis (characterization of the precision in the localization of the reflector):

• Lateral resolution: $\lambda L/a$, where λ is the central wavelength, L is the distance from the array to the reflector, and a is the array diameter (paraxial regime $\lambda \ll a \ll L$).

• Range resolution: c_0/B, where c_0 is the background velocity and B is the bandwidth.

Kirchhoff Migration:

$$\mathcal{I}_{\mathrm{KM}}(\vec{\boldsymbol{y}}^S) = \sum_{r=1}^{N_{\mathrm{r}}} \sum_{s=1}^{N_{\mathrm{s}}} u\big(\mathcal{T}(\vec{\boldsymbol{x}}_s, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{y}}^S, \vec{\boldsymbol{x}}_r), \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s\big)$$

- Resolution analysis (characterization of the precision in the localization of the reflector):
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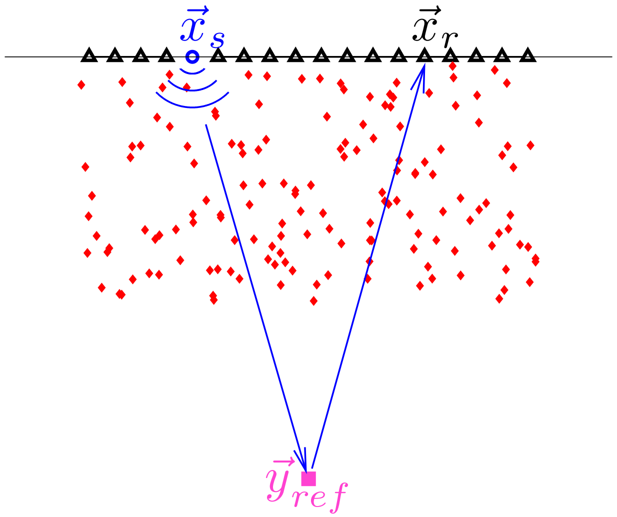
- Range resolution: c_0/B, where c_0 is the background velocity and B is the bandwidth.

- Stability analysis:
- Very robust with respect to additive measurement noise [1].
- Sensitive to medium noise: If the medium is scattering, then Kirchhoff Migration (usually) does not work.

[1] H. Ammari, J. Garnier, and K. Sølna *Waves in Random and Complex Media* **22**, 40 (2012).

Imaging through a scattering medium



• Sensor array imaging of a reflector located at $\vec{\boldsymbol{y}}_{\rm ref}$. $\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_r$ is a receiver.
Data: $\{u(t,\vec{\boldsymbol{x}}_r;\vec{\boldsymbol{x}}_s), r=1,\ldots,N_{\rm r}, s=1,\ldots,N_{\rm s}\}$.

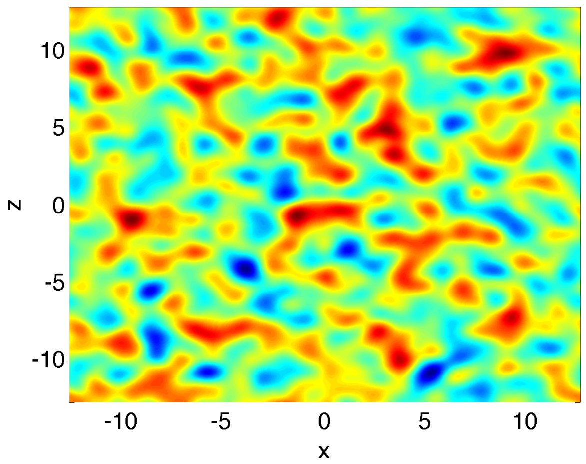
$$\Big(\frac{1}{c^2(\vec{\boldsymbol{x}})}+\frac{1}{c_{\rm ref}^2}\mathbf{1}_{B_{\rm ref}}(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}}_{\rm ref})\Big)\frac{\partial^2 u}{\partial t^2}(t,\vec{\boldsymbol{x}};\vec{\boldsymbol{x}}_{\rm s})-\Delta_{\vec{\boldsymbol{x}}}u(t,\vec{\boldsymbol{x}};\vec{\boldsymbol{x}}_{\rm s})=f(t)\delta(\vec{\boldsymbol{x}}-\vec{\boldsymbol{x}}_{\rm s})$$

• Random medium model:

$$\frac{1}{c^2(\vec{\boldsymbol{x}})}=\frac{1}{c_0^2}\big(1+\mu(\vec{\boldsymbol{x}})\big)$$

c_0 is a reference speed,

$\mu(\vec{\boldsymbol{x}})$ is a zero-mean random process.



• Remarks about medium noise:

- the medium noise $u - u_0$ (where u_0 is the data that would be obtained in a homogeneous medium) is *not* an additive white noise.

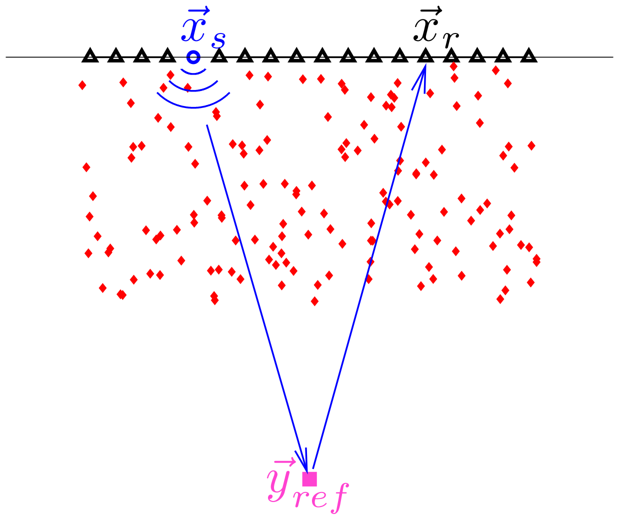
- a stochastic and multiscale analysis is possible in different regimes of separation of scales (small wavelength, large propagation distance, large bandwidth, ...).

• General results obtained by a multiscale and stochastic analysis:

• The mean (coherent) wave is small.
$\Longrightarrow$ The Kirchhoff Migration function (or Reverse Time imaging function) is unstable in randomly scattering media.

• The wave fluctuations at nearby points and nearby frequencies are correlated.
The wave correlations carry information about the medium.
$\Longrightarrow$ One should use local cross correlations (quadratic forms of the data) for imaging.

Imaging through a scattering medium

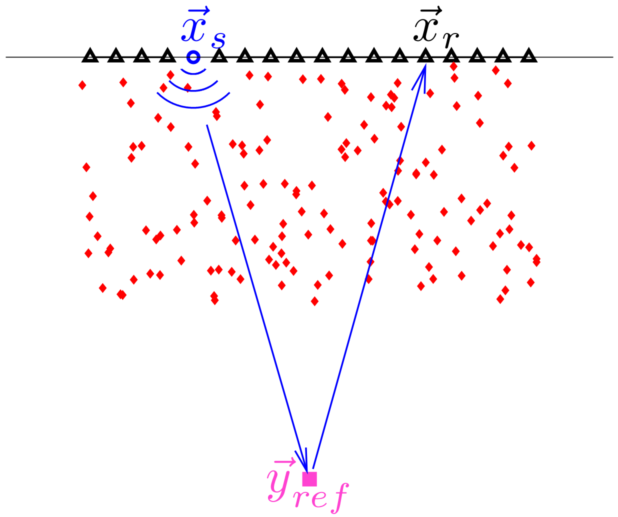


Sensor array imaging of a reflector located at $\vec{\boldsymbol{y}}_{\rm ref}$. $\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_r$ is a receiver.
Data: $\{u(t, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s), r = 1, \ldots, N_{\rm r}, s = 1, \ldots, N_{\rm s}\}$.

If the medium is scattering, then Kirchhoff migration does not work:

$$\mathcal{I}_{\rm KM}(\vec{\boldsymbol{y}}^S) = \sum_{s=1}^{N_{\rm s}} \sum_{r=1}^{N_{\rm r}} u\big(\mathcal{T}(\vec{\boldsymbol{x}}_s, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{y}}^S, \vec{\boldsymbol{x}}_r), \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s\big)$$

Imaging through a scattering medium



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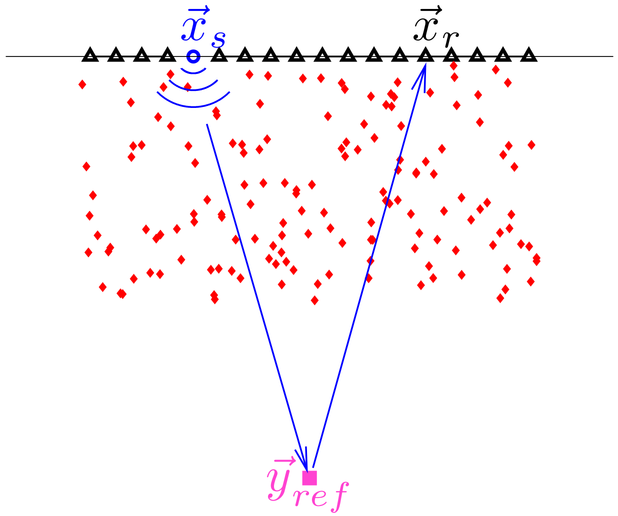
Data: $\{u(t,\vec{\boldsymbol{x}}_r;\vec{\boldsymbol{x}}_s), r=1,\dots,N_{\rm r}, s=1,\dots,N_{\rm s}\}$.

If the medium is scattering, then Kirchhoff migration does not work:

$$\mathcal{I}_{\rm KM}(\vec{\boldsymbol{y}}^S) = \sum_{s=1}^{N_{\rm s}}\sum_{r=1}^{N_{\rm r}}\int \overline{\hat{u}(\omega,\vec{\boldsymbol{x}}_r;\vec{\boldsymbol{x}}_s)}\exp\Big\{i\omega\big[\mathcal{T}(\vec{\boldsymbol{x}}_s,\vec{\boldsymbol{y}}^S)+\mathcal{T}(\vec{\boldsymbol{y}}^S,\vec{\boldsymbol{x}}_r)\big]\Big\}d\omega$$

Imaging through a scattering medium



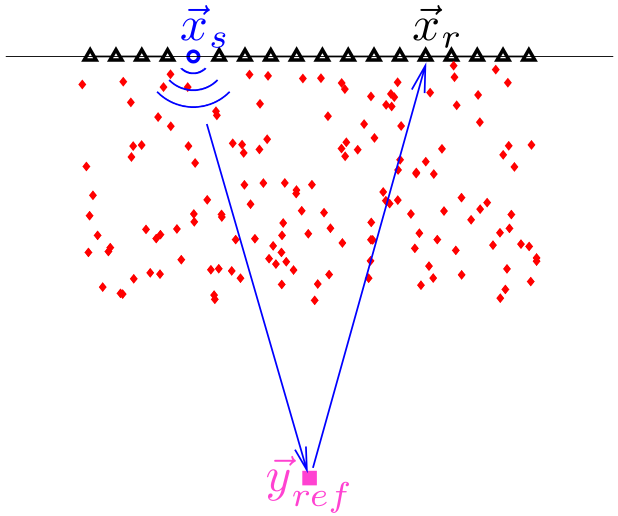
Sensor array imaging of a reflector located at $\vec{\boldsymbol{y}}_{\rm ref}$. $\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_r$ is a receiver.
Data: $\{u(t,\vec{\boldsymbol{x}}_r;\vec{\boldsymbol{x}}_s), r=1,\dots,N_{\rm r}, s=1,\dots,N_{\rm s}\}$.

If the medium is scattering, then full migration of cross correlations does not work:

$$\begin{aligned}
\mathcal{I}_{\rm fullCC}(\vec{\boldsymbol{y}}^S) &= \sum_{s,s'=1}^{N_{\rm s}} \sum_{r,r'=1}^{N_{\rm r}} \iint d\omega d\omega' \, \hat{u}(\omega,\vec{\boldsymbol{x}}_r;\vec{\boldsymbol{x}}_s)\overline{\hat{u}(\omega',\vec{\boldsymbol{x}}_{r'};\vec{\boldsymbol{x}}_{s'})} \\
&\quad \times \exp\Big\{ -i\omega\big[\mathcal{T}(\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{y}}^S)+\mathcal{T}(\vec{\boldsymbol{x}}_s,\vec{\boldsymbol{y}}^S)\big] + i\omega'\big[\mathcal{T}(\vec{\boldsymbol{x}}_{r'},\vec{\boldsymbol{y}}^S)+\mathcal{T}(\vec{\boldsymbol{x}}_{s'},\vec{\boldsymbol{y}}^S)\big]\Big\} \\
&= \Big| \sum_{s=1}^{N_{\rm s}} \sum_{r=1}^{N_{\rm r}} \int \overline{\hat{u}(\omega,\vec{\boldsymbol{x}}_r;\vec{\boldsymbol{x}}_s)} \exp\Big\{ i\omega\big[\mathcal{T}(\vec{\boldsymbol{x}}_s,\vec{\boldsymbol{y}}^S)+\mathcal{T}(\vec{\boldsymbol{y}}^S,\vec{\boldsymbol{x}}_r)\big]\Big\} d\omega \Big|^2 = \big|\mathcal{I}_{\rm KM}(\vec{\boldsymbol{y}}^S)\big|^2
\end{aligned}$$

If one migrates all cross correlations, one gets the same image as with Kirchhoff !

Imaging through a scattering medium



Sensor array imaging of a reflector located at $\vec{\boldsymbol{y}}_{\rm ref}$. $\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_r$ is a receiver.
Data: $\{u(t, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s), r = 1, \ldots, N_{\rm r}, s = 1, \ldots, N_{\rm s}\}$.

If the medium is scattering, then use Coherent Interferometric Imaging (CINT):

$$\mathcal{I}_{\rm CINT}(\vec{\boldsymbol{y}}^S) = \sum_{\substack{s,s'=1 \\ |\vec{\boldsymbol{x}}_s - \vec{\boldsymbol{x}}_{s'}| \leq X_{\rm d}}}^{N_{\rm s}} \sum_{\substack{r,r'=1 \\ |\vec{\boldsymbol{x}}_r - \vec{\boldsymbol{x}}_{r'}| \leq X_{\rm d}}}^{N_{\rm r}} \iint_{|\omega - \omega'| \leq \Omega_{\rm d}} d\omega d\omega' \, \hat{u}(\omega, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s) \overline{\hat{u}(\omega', \vec{\boldsymbol{x}}_{r'}; \vec{\boldsymbol{x}}_{s'})}$$
$$\times \exp\Big\{ - i\omega \big[\mathcal{T}(\vec{\boldsymbol{x}}_r, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{x}}_s, \vec{\boldsymbol{y}}^S)\big] + i\omega' \big[\mathcal{T}(\vec{\boldsymbol{x}}_{r'}, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{x}}_{s'}, \vec{\boldsymbol{y}}^S)\big] \Big\}$$

It forms the image with the superposition of the backpropagated local cross correlations of the traces.

Coherent Interferometric Imaging (CINT):

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$$\times \exp\Big\{ -i\omega \big[\mathcal{T}(\vec{\boldsymbol{x}}_r, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{x}}_s, \vec{\boldsymbol{y}}^S)\big] + i\omega' \big[\mathcal{T}(\vec{\boldsymbol{x}}_{r'}, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{x}}_{s'}, \vec{\boldsymbol{y}}^S)\big] \Big\}$$

- Resolution analysis:

Lateral resolution: $\lambda L / X_{\rm d}$ (for $X_{\rm d} < a$, where a is the array diameter).

Range resolution: $c_0/\Omega_{\rm d}$ (for $\Omega_{\rm d} < B$, where B is the bandwidth).

[1] L. Borcea, J. Garnier, G. Papanicolaou, and C. Tsogka, *Inverse Problems* **27**, 085004 (2011).

Coherent Interferometric Imaging (CINT):

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Range resolution: $c_0/\Omega_{\rm d}$ (for $\Omega_{\rm d} < B$, where B is the bandwidth).

- Statistical stability:

$$\mathrm{SNR}_{\rm CINT} := \frac{\mathbb{E}\big[\mathcal{I}_{\rm CINT}(\vec{\boldsymbol{y}}^S)\big]}{\mathrm{Var}\big(\mathcal{I}_{\rm CINT}(\vec{\boldsymbol{y}}^S)\big)^{1/2}} > 1 \text{ when } \frac{X_{\rm d}}{X_{\rm c}} < 1, \ \frac{a}{X_{\rm c}} > 1 \text{ and/or } \frac{\Omega_{\rm d}}{\Omega_c} < 1, \ \frac{B}{\Omega_{\rm c}} > 1$$

where $X_{\rm c}$ is the decoherence length (distance between sensors beyond which the signals are not correlated) and $\Omega_{\rm c}$ is the decoherence frequency (frequency gap beyond which the frequency components of the recorded signals are not correlated).

[1] L. Borcea, J. Garnier, G. Papanicolaou, and C. Tsogka, *Inverse Problems* **27**, 085004 (2011).

Coherent Interferometric Imaging (CINT):

$$\mathcal{I}_{\rm CINT}(\vec{\boldsymbol{y}}^S) = \sum_{\substack{s,s'=1 \\ |\vec{\boldsymbol{x}}_s - \vec{\boldsymbol{x}}_{s'}| \leq X_{\rm d}}}^{N_{\rm s}} \sum_{\substack{r,r'=1 \\ |\vec{\boldsymbol{x}}_r - \vec{\boldsymbol{x}}_{r'}| \leq X_{\rm d}}}^{N_{\rm r}} \iint_{|\omega-\omega'|\leq \Omega_{\rm d}} d\omega d\omega' \, \hat{u}(\omega, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s) \overline{\hat{u}(\omega', \vec{\boldsymbol{x}}_{r'}, \vec{\boldsymbol{x}}_{s'})}$$
$$\times \exp\Big\{ -i\omega\big[\mathcal{T}(\vec{\boldsymbol{x}}_r, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{x}}_s, \vec{\boldsymbol{y}}^S)\big] + i\omega'\big[\mathcal{T}(\vec{\boldsymbol{x}}_{r'}, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{x}}_{s'}, \vec{\boldsymbol{y}}^S)\big]\Big\}$$

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Lateral resolution: $\lambda L / X_{\rm d}$ (for $X_{\rm d} < a$, where a is the array diameter).
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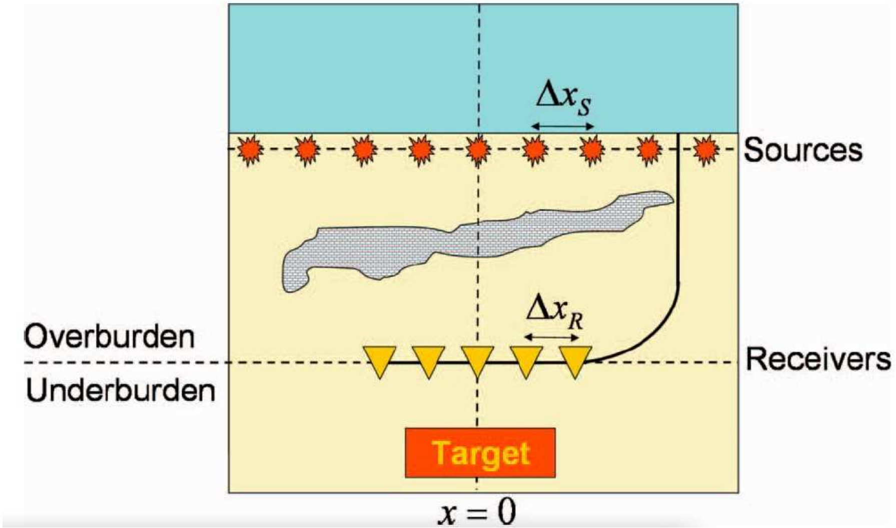
where $X_{\rm c}$ is the decoherence length (distance between sensors beyond which the signals are not correlated) and $\Omega_{\rm c}$ is the decoherence frequency (frequency gap beyond which the frequency components of the recorded signals are not correlated).

- Optimal values $\Omega_{\rm d} = \Omega_{\rm c}$ and $X_{\rm d} = X_{\rm c}$. They can be determined by

\- a statistical analysis of the data.
\- an adaptive procedure minimizing a suitable norm of the image.
\- a good a priori choice !

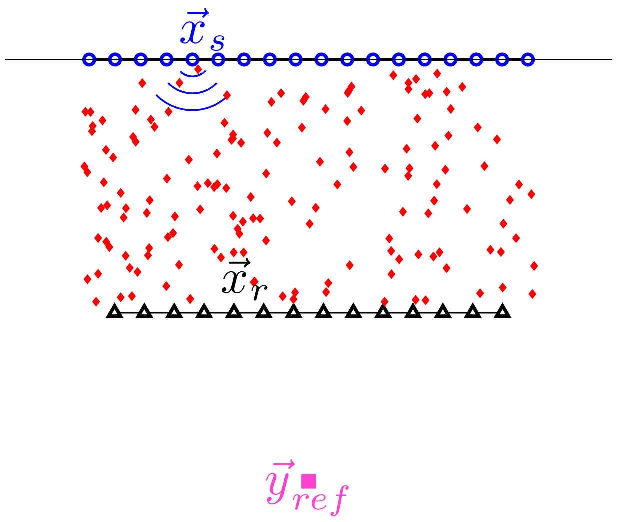
[1] L. Borcea, J. Garnier, G. Papanicolaou, and C. Tsogka, *Inverse Problems* **27**, 085004 (2011).

Imaging below an "overburden"



From van der Neut and Bakulin (2009)

Imaging below an overburden

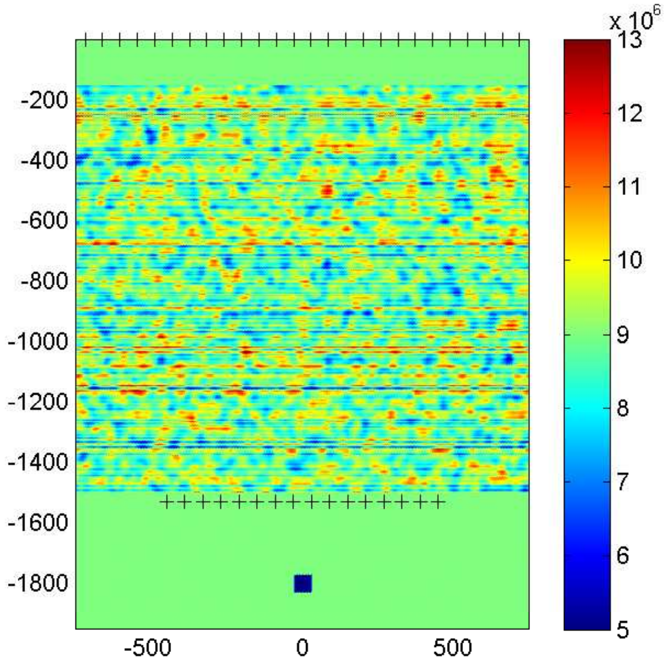


Array imaging of a reflector at $\vec{\boldsymbol{y}}_{\rm ref}$. $\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_r$ is a receiver located below the scattering medium. Data: $\{u(t, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s), r = 1, \ldots, N_{\rm r}, s = 1, \ldots, N_{\rm s}\}$.

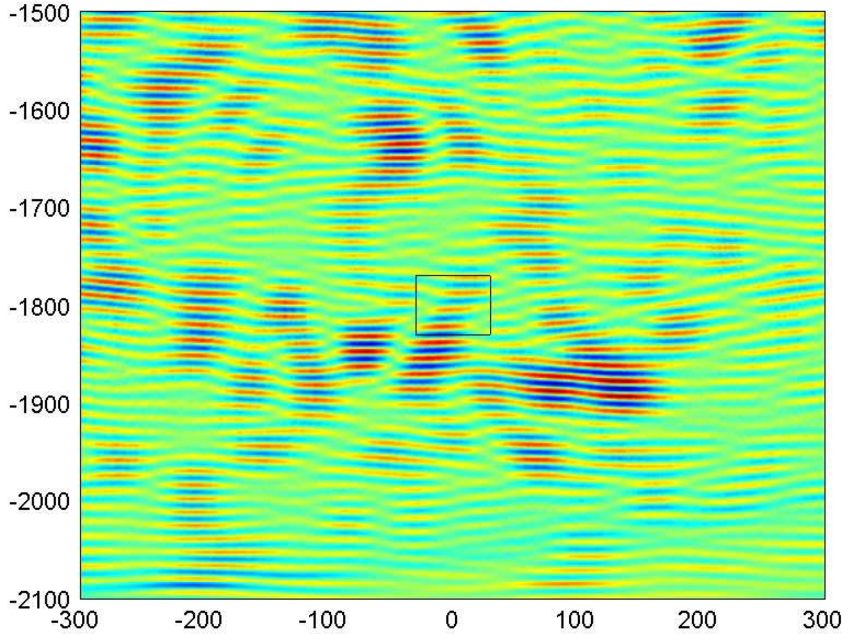
If the overburden is scattering, then Kirchhoff Migration does not work:

$$\mathcal{I}_{\rm KM}(\vec{\boldsymbol{y}}^S) = \sum_{r=1}^{N_{\rm r}} \sum_{s=1}^{N_{\rm s}} u\big(\mathcal{T}(\vec{\boldsymbol{x}}_s, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{y}}^S, \vec{\boldsymbol{x}}_r), \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s\big)$$

Numerical simulations

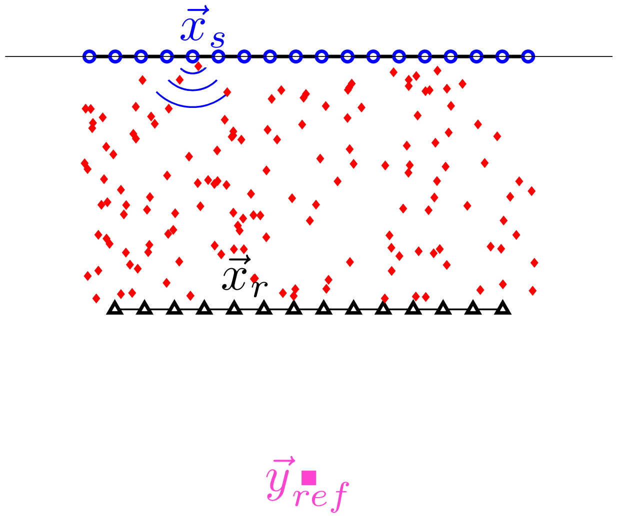


Computational setup



Kirchhoff Migration

Imaging below an overburden



$\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_r$ is a receiver. Data: $\{u(t,\vec{\boldsymbol{x}}_r;\vec{\boldsymbol{x}}_s), r=1,\ldots,N_\mathrm{r}, s=1,\ldots,N_\mathrm{s}\}$.

Image with migration of the special cross correlation matrix:

$$\mathcal{I}(\vec{\boldsymbol{y}}^S) = \sum_{r,r'=1}^{N_\mathrm{r}} \mathcal{C}\big(\mathcal{T}(\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{y}}^S)+\mathcal{T}(\vec{\boldsymbol{y}}^S,\vec{\boldsymbol{x}}_{r'}),\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{x}}_{r'}\big),$$

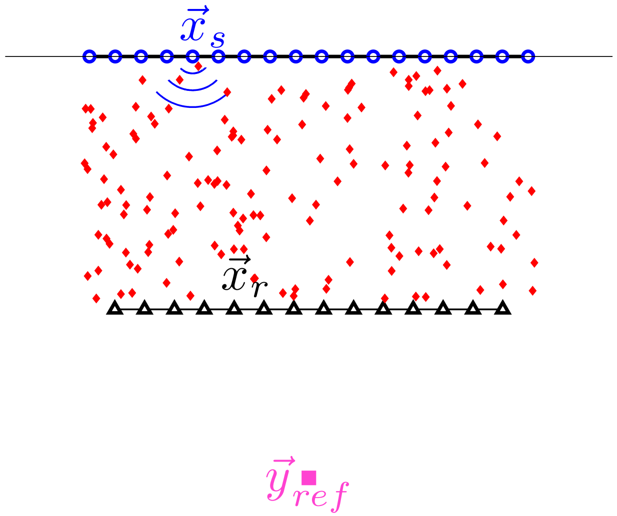
with

$$\mathcal{C}(\tau,\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{x}}_{r'}) = \sum_{s=1}^{N_\mathrm{s}} \int u(t,\vec{\boldsymbol{x}}_r;\vec{\boldsymbol{x}}_s)u(t+\tau,\vec{\boldsymbol{x}}_{r'};\vec{\boldsymbol{x}}_s)dt\ , \qquad r,r'=1,\ldots,N_\mathrm{r}$$

Related to CINT, imaging with ambient noise sources, virtual source method [1].

[1] A. Bakulin and R. Calvert, *Geophysics* **71** (2006), SI139-SI150.

Imaging below an overburden



$\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_r$ is a receiver. Data: $\{u(t, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s), r = 1, \ldots, N_\mathrm{r}, s = 1, \ldots, N_\mathrm{s}\}$.

Image with migration of the special cross correlation matrix:

$$\mathcal{I}(\vec{\boldsymbol{y}}^S) = \sum_{r,r'=1}^{N_\mathrm{r}} \mathcal{C}\big(\mathcal{T}(\vec{\boldsymbol{x}}_r, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{y}}^S, \vec{\boldsymbol{x}}_{r'}), \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_{r'}\big),$$

with

$$\mathcal{C}(\tau, \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_{r'}) = \sum_{s=1}^{N_\mathrm{s}} \int u(t, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s) u(t+\tau, \vec{\boldsymbol{x}}_{r'}; \vec{\boldsymbol{x}}_s) dt \ , \qquad r, r' = 1, \ldots, N_\mathrm{r}$$

It is a special CINT function:

$$\mathcal{I}(\vec{\boldsymbol{y}}^S) = \frac{1}{2\pi} \sum_{s=1}^{N_\mathrm{s}} \sum_{r,r'=1}^{N_\mathrm{r}} \int d\omega \hat{u}(\omega, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s) \overline{\hat{u}(\omega, \vec{\boldsymbol{x}}_{r'}; \vec{\boldsymbol{x}}_s)} \exp\Big\{ i\omega \big[\mathcal{T}(\vec{\boldsymbol{x}}_r, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{y}}^S, \vec{\boldsymbol{x}}_{r'}) \big] \Big\}$$

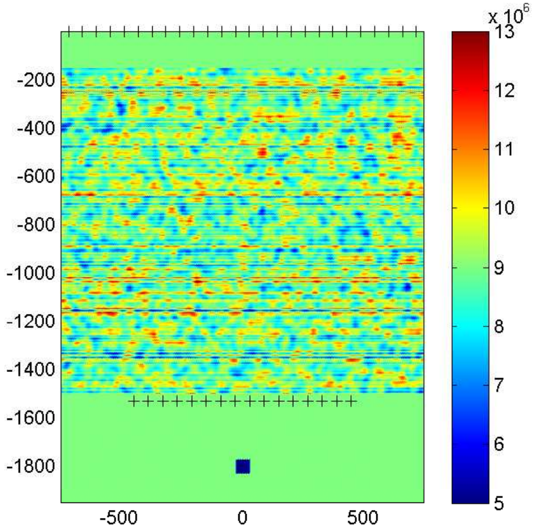
Remark: General CINT function:

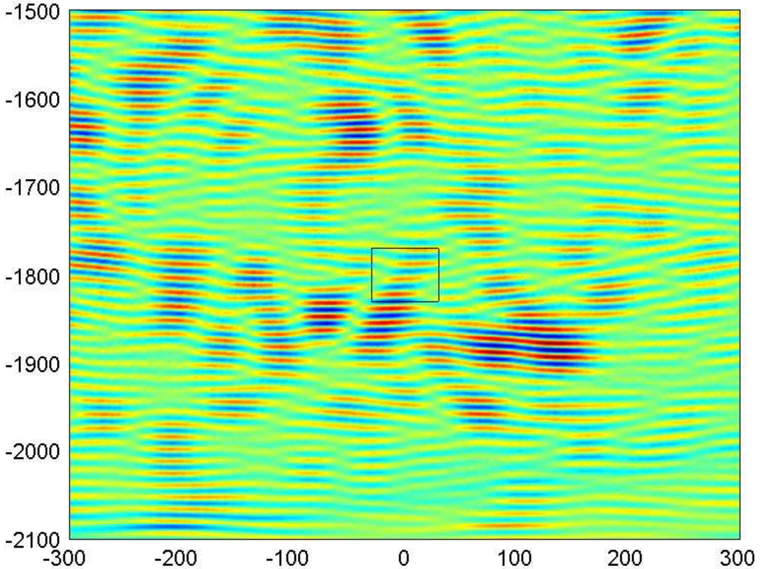
$$\mathcal{I}_{\rm CINT}(\vec{\boldsymbol{y}}^S) = \sum_{\substack{s,s'=1 \\ |\vec{\boldsymbol{x}}_s-\vec{\boldsymbol{x}}_{s'}|\leq X_{\rm d}}}^{N_{\rm s}} \sum_{\substack{r,r'=1 \\ |\vec{\boldsymbol{x}}_r-\vec{\boldsymbol{x}}_{r'}|\leq X'_{\rm d}}}^{N_{\rm r}} \iint_{|\omega-\omega'|\leq\Omega_{\rm d}} d\omega d\omega' \, \hat{u}(\omega,\vec{\boldsymbol{x}}_r;\vec{\boldsymbol{x}}_s)\overline{\hat{u}(\omega',\vec{\boldsymbol{x}}_{r'},\vec{\boldsymbol{x}}_{s'})}$$

$$\times \exp\Big\{ - i\omega\big[\mathcal{T}(\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{x}}_s,\vec{\boldsymbol{y}}^S)\big] + i\omega'\big[\mathcal{T}(\vec{\boldsymbol{x}}_{r'},\vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{x}}_{s'},\vec{\boldsymbol{y}}^S)\big]\Big\}$$

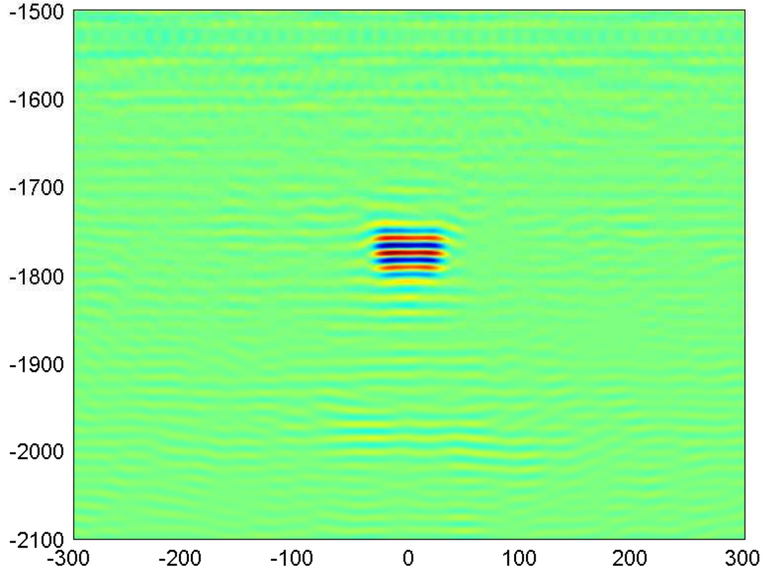
- If $X_{\rm d} = X'_{\rm d} = \Omega_{\rm d} = \infty$, then $\mathcal{I}_{\rm CINT}(\vec{\boldsymbol{y}}^S) = \big|\mathcal{I}_{\rm KM}(\vec{\boldsymbol{y}}^S)\big|^2$.
- If $X_{\rm d} = 0$, $X'_{\rm d} = \infty$, $\Omega_{\rm d} = 0$, then $\mathcal{I}_{\rm CINT}(\vec{\boldsymbol{y}}^S)$ is the special CINT.

Numerical simulations





Kirchhoff Migration



Cross Correlation Migration

Analysis in randomly scattering media

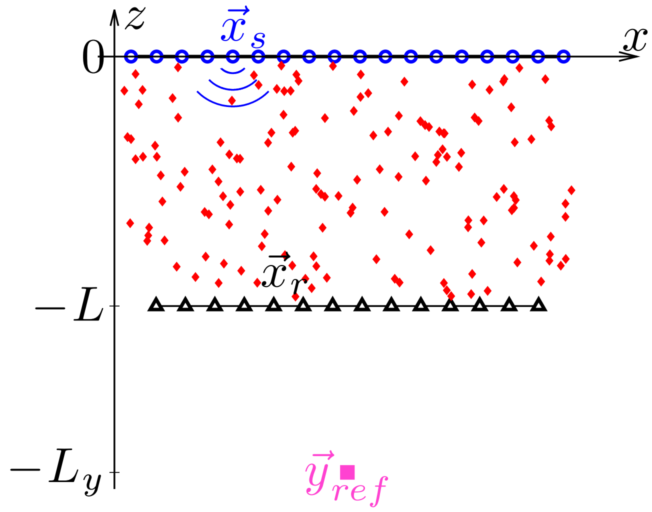
• Does the cross correlation imaging function give good images in scattering media ?
↪ It is possible to analyze the resolution and stability of the imaging function in randomly scattering media.

• General results:
Imaging function is stable provided the bandwidth is large enough and/or the source array is large enough.

• Detailed results:
If there are sources everywhere at the surface: scattering plays no role.
If the source distribution is spatially limited: scattering is important.
- in the random paraxial regime, scattering helps (it enhances the angular diversity of the illumination).
- in the randomly layered regime, scattering does not help (it reduces the angular diversity of the illumination).

[1] J. Garnier and G. Papanicolaou, *Inverse Problems* **28** 075002 (2012).

Imaging below an overburden: analysis in the paraxial regime

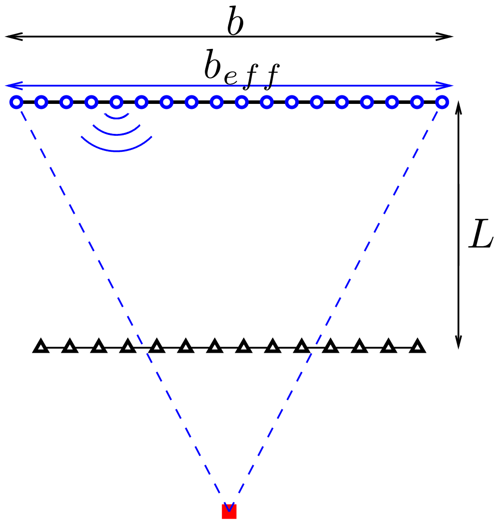


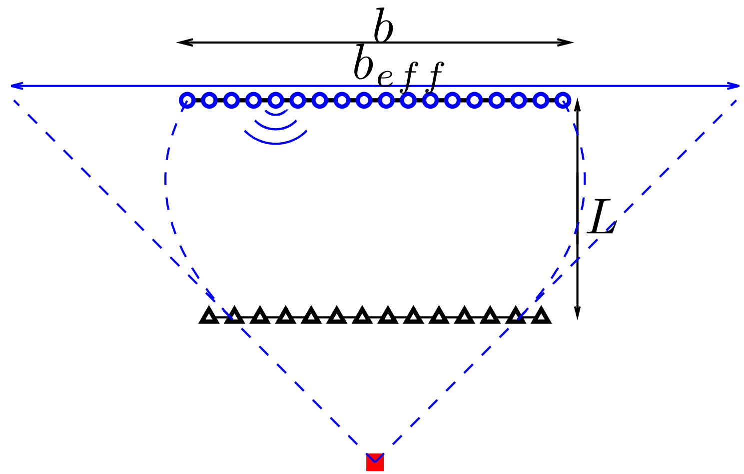
- Consider the regime "$\lambda \ll l_c \ll L$".

- Assume that:

- the source aperture is b and the receiver aperture is a.
- there is a point reflector at $\vec{\boldsymbol{y}}_{\rm ref} = (\boldsymbol{y}, -L_y)$.
- the covariance function $\gamma(\boldsymbol{x}) = \int \mathbb{E}[\mu(\boldsymbol{0},0)\mu(\boldsymbol{x},z)]dz$ can be expanded as $\gamma(\boldsymbol{x}) = \gamma(\boldsymbol{0}) - \bar{\gamma}_2|\boldsymbol{x}|^2 + o(|\boldsymbol{x}|^2)$ for small $|\boldsymbol{x}|$.
- scattering is strong: $\frac{\gamma(\boldsymbol{0})\omega_0^2 L}{c_0^2} > 1$ ($\to$ mean wave is damped).

Imaging below an overburden: analysis in the paraxial regime





Homogeneous medium

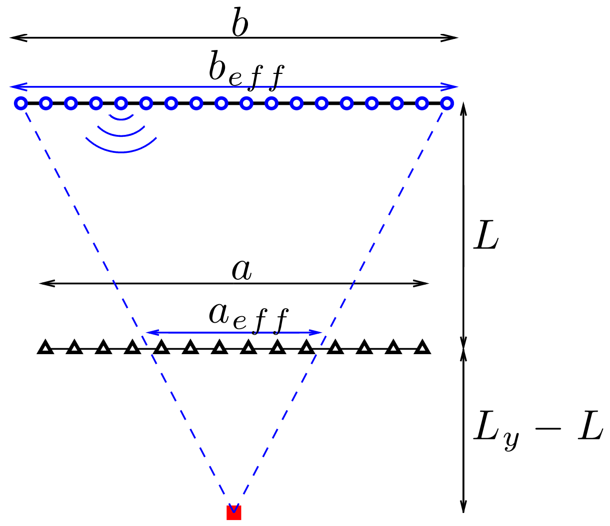
Random medium

Effective source aperture:

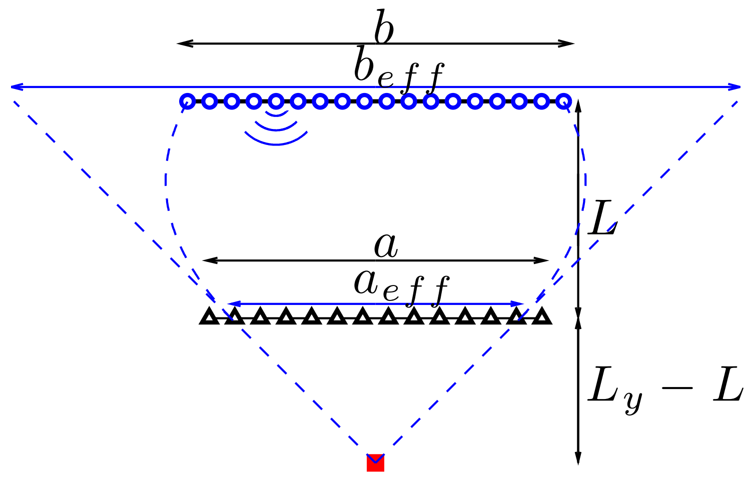
$$b_{\text{eff}} = b$$

$$b_{\text{eff}} = \Big(b^2 + \frac{\bar{\gamma}_2 L^3}{3}\Big)^{1/2}$$

Imaging below an overburden: analysis in the paraxial regime



Homogeneous medium



Random medium

Effective source aperture:

$$b_{\text{eff}} = b \qquad\qquad b_{\text{eff}} = \Big(b^2 + \frac{\bar{\gamma}_2 L^3}{3}\Big)^{1/2}$$

Effective receiver aperture:

$$a_{\text{eff}} = b\frac{L_y - L}{L_y} \qquad\qquad a_{\text{eff}} = b_{\text{eff}}\frac{L_y - L}{L_y}$$

Imaging below an overburden: analysis in the paraxial regime

- The imaging function for the search point $\vec{\boldsymbol{y}}^S$ is

$$\mathcal{I}(\vec{\boldsymbol{y}}^S) = \frac{1}{N_{\rm r}^2} \sum_{r,r'=1}^{N_{\rm r}} \mathcal{C}\big(\mathcal{T}(\vec{\boldsymbol{x}}_r, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{y}}^S, \vec{\boldsymbol{x}}_{r'}), \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_{r'}\big)$$

- The imaging function is statistically stable ($\lambda_0 \ll b \ll L$).

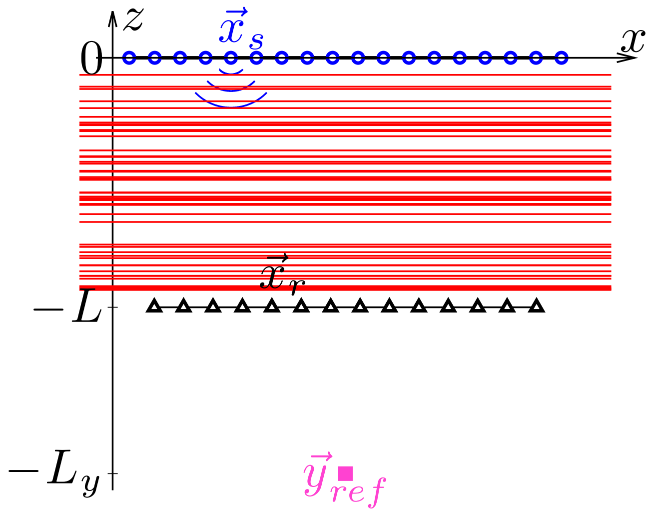
- The lateral resolution is $\dfrac{\lambda_0(L_y - L)}{a_{\rm eff}}$. The range resolution is $\dfrac{c_0}{B}$.

Here: λ_0 is the carrier wavelength, B is the bandwidth.

- Since $a_{\rm eff} \mid_{\rm rand} > a_{\rm eff} \mid_{\rm homo}$, this shows that scattering helps.

- physical reason: scattering enhances the angular diversity of the illumination.

Imaging below an overburden: analysis in the layered regime



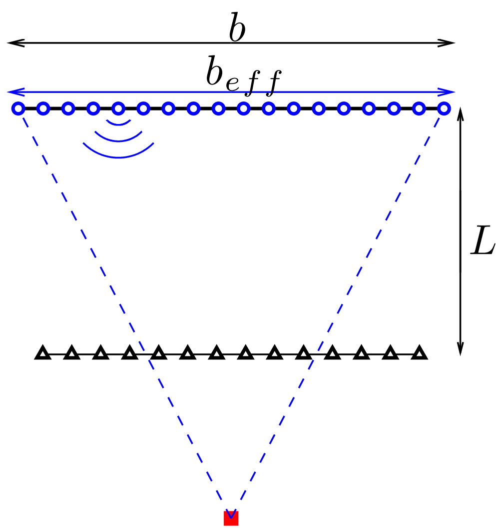
- Consider the regime "$l_c \ll \lambda \ll L$".

- Assume that:

- the source aperture is b and the receiver aperture is a.
- there is a point reflector at $\vec{\boldsymbol{y}}_{\rm ref} = (\boldsymbol{y}, -L_y)$.
- the localization length $L_{\rm loc}$ is smaller than L (strong scattering, mean wave is damped):

$$L_{\rm loc} = \frac{4c_0^2}{\gamma\omega_0^2}, \qquad \gamma = \int_{-\infty}^{\infty} \mathbb{E}[\mu(0)\mu(z)]dz$$

Imaging below an overburden: analysis in the layered regime



Homogeneous medium

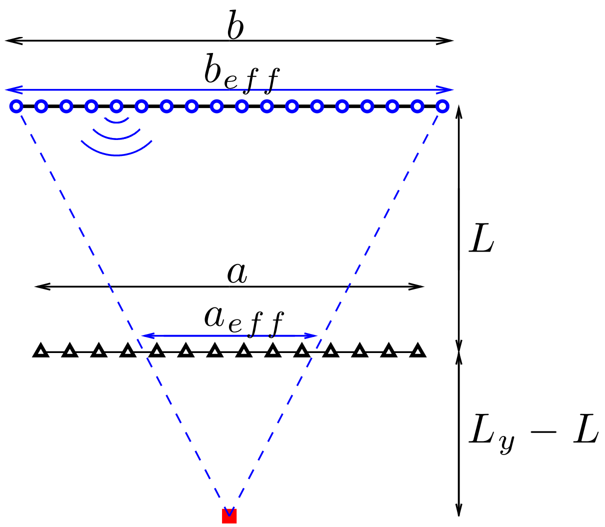
Randomly layered medium

Effective source aperture:

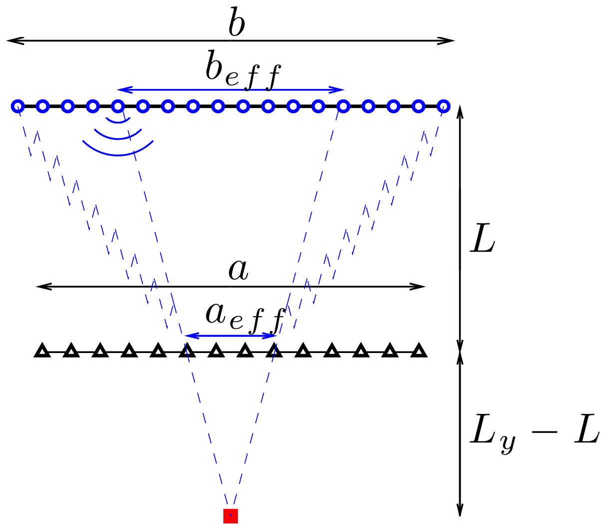
$b_{\text{eff}} = b$

$b_{\text{eff}}^2 = 4 L_{\text{loc}} L \ (\ll b^2)$

Imaging below an overburden: analysis in the layered regime



Homogeneous medium



Randomly layered medium

Effective source aperture:

$b_{\text{eff}} = b$

$b_{\text{eff}}^2 = 4L_{\text{loc}}L \;\; (\ll b^2)$

Effective receiver aperture:

$$a_{\text{eff}} = b\frac{L_y - L}{L_y}$$

$$a_{\text{eff}} = b_{\text{eff}}\frac{L_y - L}{L_y}$$

Imaging below an overburden: analysis in the layered regime

- The imaging function for the search point $\vec{\boldsymbol{y}}^S$ is

$$\mathcal{I}(\vec{\boldsymbol{y}}^S) = \frac{1}{N_{\rm r}^2} \sum_{r,r'=1}^{N_{\rm r}} \mathcal{C}\big(\mathcal{T}(\vec{\boldsymbol{x}}_r, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{y}}^S, \vec{\boldsymbol{x}}_{r'}), \vec{\boldsymbol{x}}_r, \vec{\boldsymbol{x}}_{r'}\big)$$

- The imaging function is statistically stable ($\lambda_0 \ll b, L$).

- The lateral resolution is $\dfrac{\lambda_0(L_y - L)}{a_{\rm eff}}$. The range resolution is $\dfrac{c_0}{B}\big(1 + \dfrac{B^2 L}{4\omega_0^2 L_{\rm loc}}\big)^{1/2}$.

- Since $a_{\rm eff}\ |_{\rm rand} < a_{\rm eff}\ |_{\rm homo}$, this shows that scattering does not help.

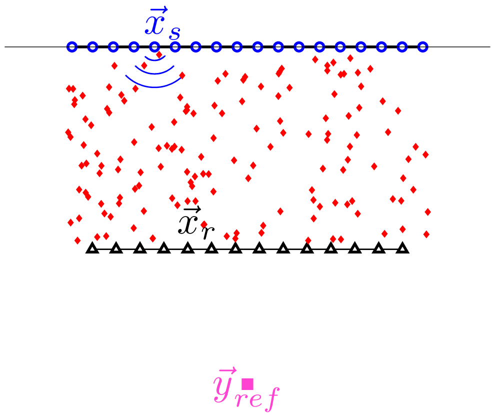
\- physical reason: scattering reduces the angular and frequency diversity of the illumination.

Conclusions

• In randomly scattering media it is better to migrate *well-chosen* cross correlations of data than data themselves.

• Correlation-based imaging can be used with ambient noise sources.

• Good situation for the cross correlation technique (with active sources everywhere):



• What is the role of scattering if the sources are spatially localized ?
- in the isotropic case, random scattering helps (enhances the source aperture).
- in the layered case, random scattering is bad (reduces the source aperture).

• Here the medium was assumed to be homogeneous in the underburden.
What happens if it is scattering ? Modify the cut-off parameters of the CINT functional (for weakly scattering underburden).