Coherence in chaotic coupled logistc map lattices

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Introduction - logistic map

The logistic map, defined by the iterative scheme $u_{n+1} = f(u_n)$ with the function

$$f(u) = cu(1 - u) \tag{1}$$

and a starting value u_0 taken from the interval $[0, 1]$, constitutes a most remarkable model system which has provided deep insights into nonlinear dynamics. The simplicity of this quadratic map contrasts with the richness of its dynamical behavior; for most values of the parameter c between 3.57 and 4 the sequence $\{u_n\}$ is chaotic. The famous universal constant $\delta = 4.669\ldots$ which emerges as a limit when the ratio of the length of c-intervals between consecutive period doublings is taken, has been discovered while investigating this map.

Coupled map lattices

Coupling of the above maps leads to extended dynamical systems which form a subclass of the so-called coupled map lattices (CMLs).
Their properties:

- temporal and spatial coordinates are discrete;
- dependent variable is allowed to take on continous values;
- can be viewed as generalizations of cellular automata, or as classical fields defined on a grid;
- have been used to study:
  - pattern formation out of equilibrium;
  - spatiotemporal chaos in extended dynamical systems;
  - Rayleigh-Benard convection;
  - dynamics of boiling;
  - formation and dynamics of clouds;
  - crystal growth processes;
  - hydrodynamics of two-dimensional flows.

Objective

We demonstrate that simple coupled map lattices based on logistic maps exhibit features associated with coherence, in the sharply defined sense of optics and quantum physics of coherent many-particle systems.

The model

A two-dimensional square lattice with grid points labeled (x, y), with x and y being integer numbers, and $t = 0, 1, 2, 3, \ldots$ is a discretized and dimensionless time variable. A field $\psi(x, y; t)$ which evolves in time according to:

$$\begin{aligned}\psi(x, y; t + 1) = {} & (1 - 4d) f(\psi(x, y, t)) \\ & + d\Big[f(\psi(x + 1, y; t)) + f(\psi(x - 1, y; t)) \\ & + f(\psi(x, y + 1; t)) + f(\psi(x, y - 1, t))\Big] ,\end{aligned} \tag{2}$$

Wel limit ourselves to the case of almost maximal d, that is $d \approx 0.25$.

Description of numerical experiments

- Grid points extend from $x = 0$ to $x = N_1$ in x-direction and from $y = 0$ to $y = N_2$ in y-direction, with $N_1 = 300$ and $N_2 = 250$.
- Dirichlet boundary condition $\psi = 0$ for both $y = 0$ and $y = N_2$
- Boundary value $\psi = 0.9$ its left margin $x = 0$ is enforced.
- The value of ψ at all other lattice points is initially set to zero.
- Absorbing boundary condition near $x = N_1$.
- $\psi = 0$ boundary condition at $x_{slit} = N_2/3$ for $y < N_2/5$, for $2N_2/5 < y < 3N_2/5$, and for $y > 4N_2/5$ to simulate double-slit experiment.

At $x_{screen} = 2N_2/3 = 170$ we place a "screen", that is, we collect the values of $\psi(x_{screen}, y; t) = \phi(y; t)$ at this x coordinate.

Coherence matrix

The normalized coherence matrix is used to investigate the properties of ψ:

$$G(y, y') = \frac{\langle \phi(y; t)\phi(y'; t)\rangle}{\mathrm{Tr}\left(\langle \phi(y; t)\phi(y'; t)\rangle\right)} ,$$

Results: patterns in double-slit experiments

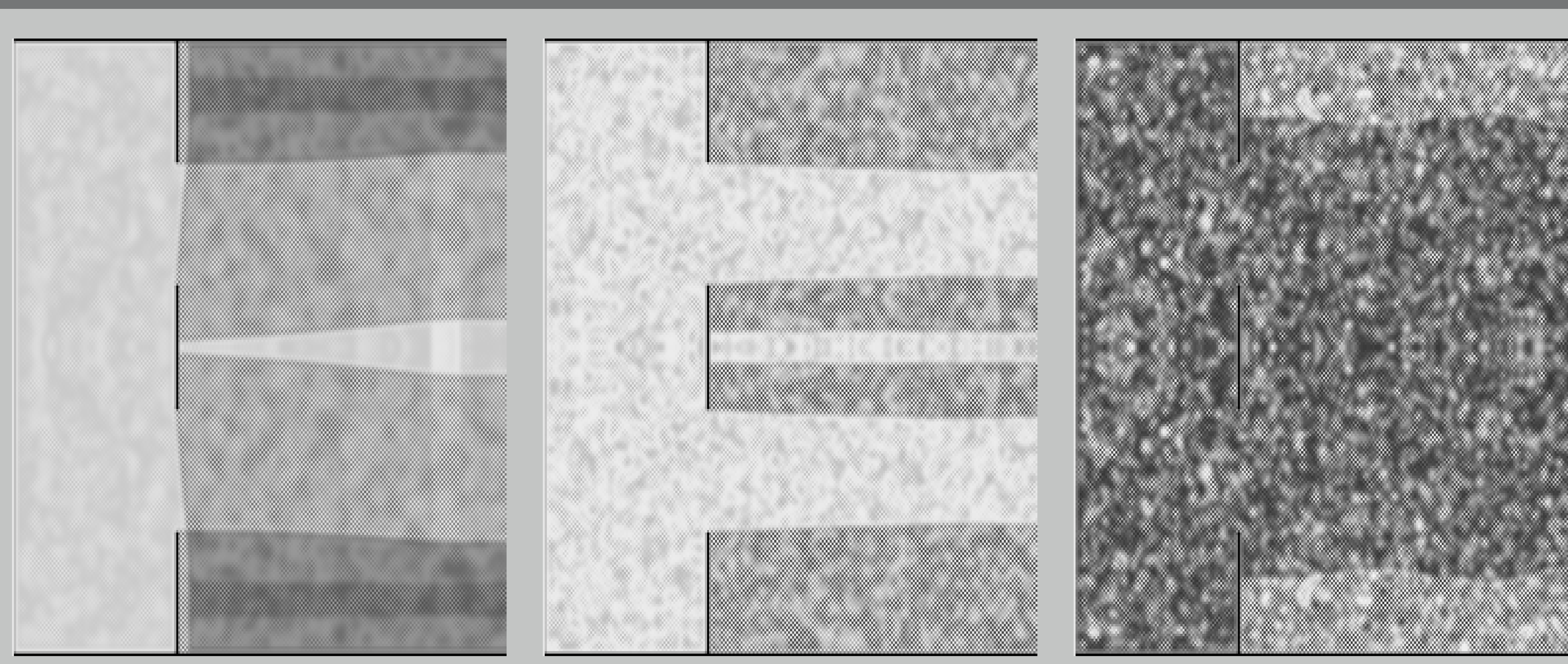

Figure: Contour plots of $I(y) = G(y; y)$ for $c = 3.6$, 3.7, and 3.8, brighter regions indicate higher values of ψ. The area covered here ranges from $x = 0$ to $x = N_2 = 250$ horizontally, and from $y = 0$ to $y = N_2$ vertically. The double slit is located at $x = N_2/3$.

Results: fringes for small nonlinear parameter

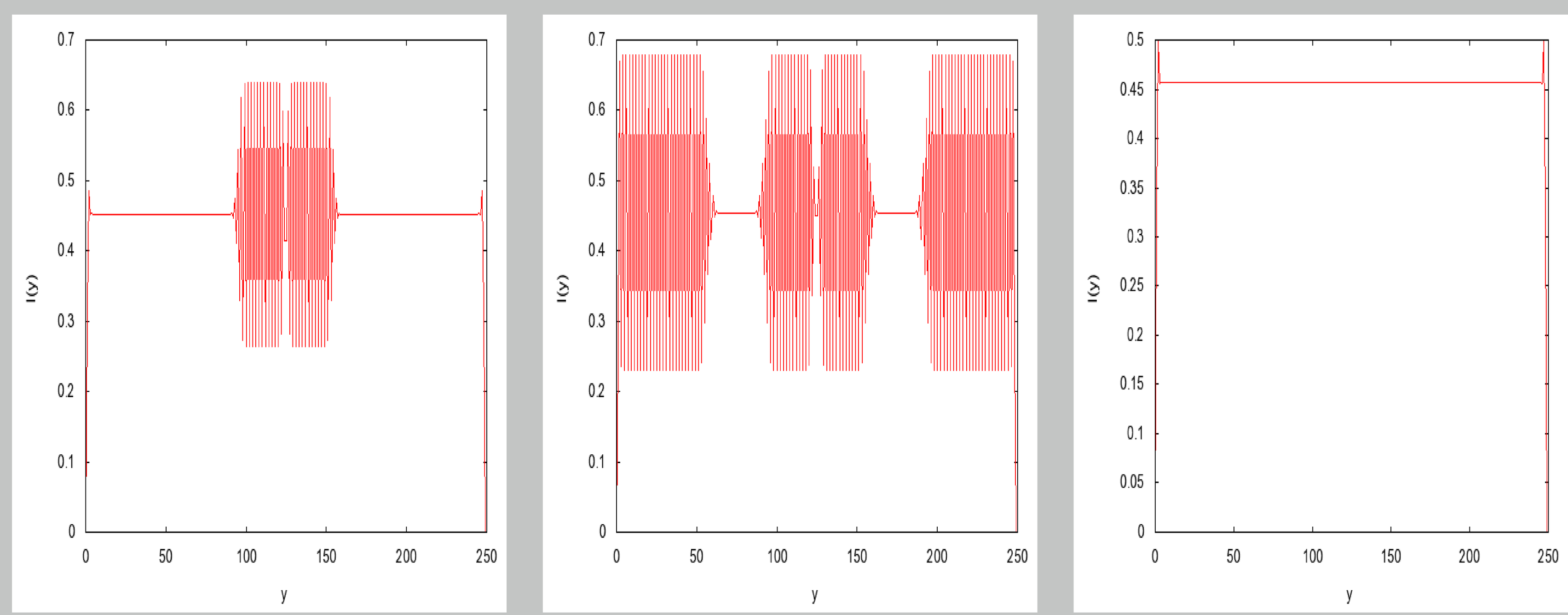

Figure: Plots of $I(y) = G(y; y)$ for $c = 3.2$, 3.3, and 3.4 for $x = N_2/3$.

Results: fringes for intermediate values of nonlinear parameter

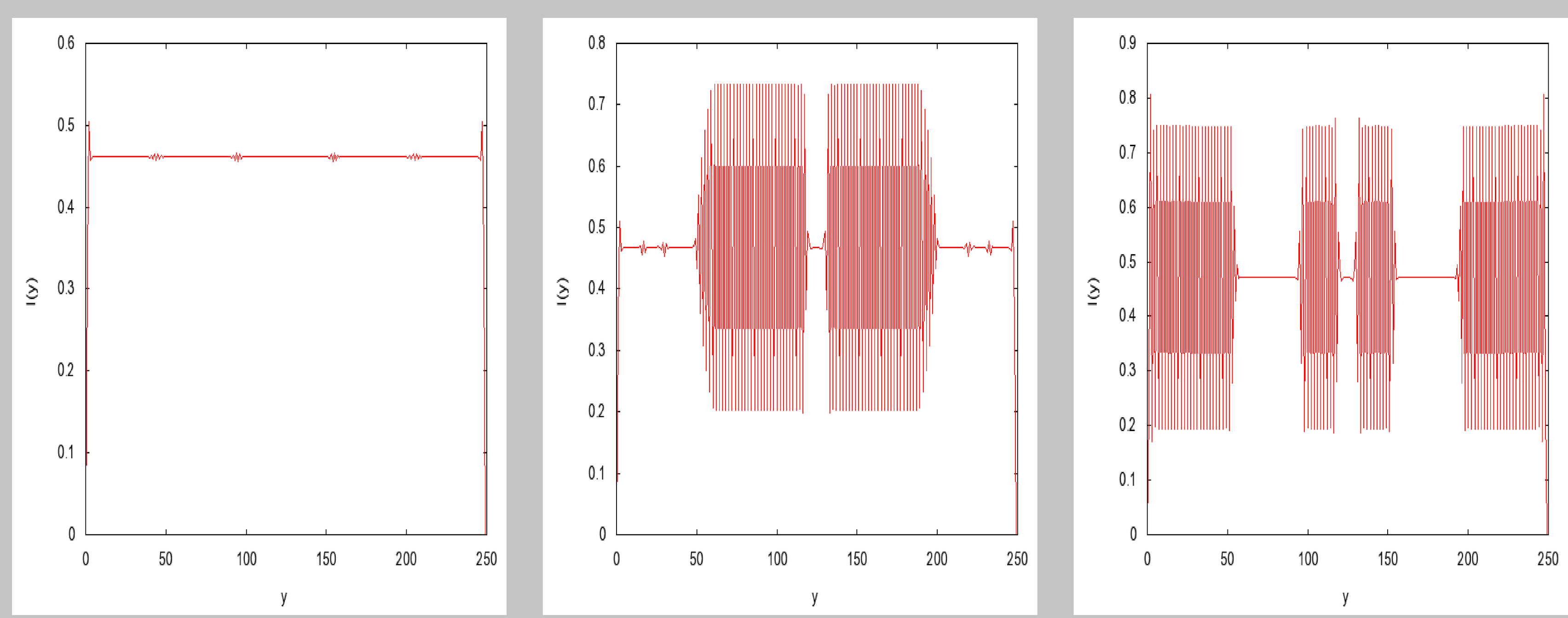

Figure: Plots of $I(y) = G(y; y)$ for $c = 3.5$, 3.6, and 3.7 for $x = N_2/3$.

Results: fringes for large nonlinear parameter

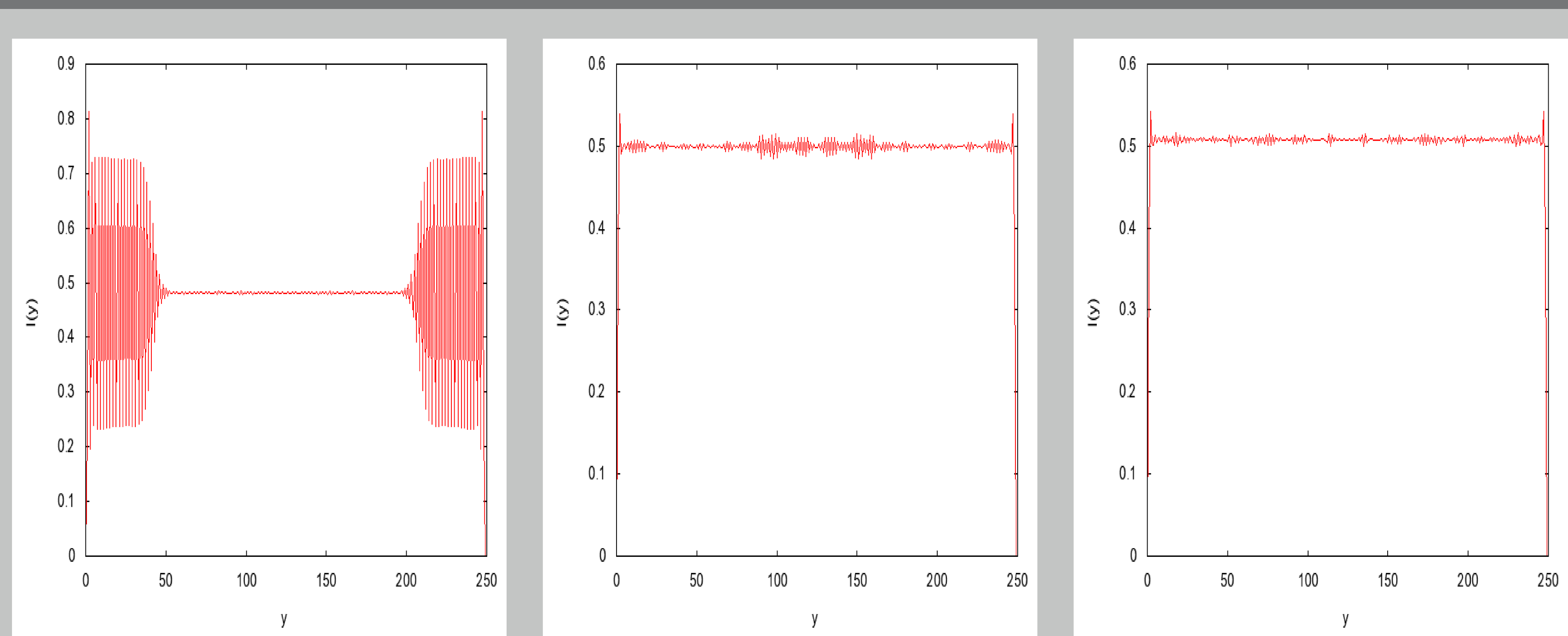

Figure: Plots of $I(y) = G(y; y)$ for $c = 3.8$, 3.9, and 4.0 for $x = N_2/3$.

Conclusions

- Notions of correlation and coherence, precisely as used in optics and quantum many-body physics, can be meaningfully applied to coupled logistic map lattices, which constitute nonlinear diffusive systems.
- Local chaos by no means obstructs genuine large-scale coherence.
- Interference patterns, not being a simple sum of diffractive patterns, can and do appear in such systems.
- *But* there are essential conditions:
  - nonlinearity must be sufficiently strong;
  - diffusion must be very fast.