Phenomenological bosonization in multicomponent systems

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Single component bosonization

One dimensional particles $\Rightarrow$ excitations become density waves.

$$H = \frac{\hbar u}{2\pi} \int dx \left[K(\pi\Pi)^2 + (\partial_x \phi)^2 \right],$$

One dimensional acoustic phonons Hamiltonian.

1. $\Pi \propto$ Momentum density $j(x)$
2. $\partial_x \phi \propto$ particle density $\rho(x)$
3. $[\phi(x), \Pi(x')] = i\delta(x - x')$ (related to $[j(x'), \rho(x)] = i\partial_x(\delta(x - x')))$
4. $u =$ velocity of excitations
5. $K =$ Luttinger exponent

$\longrightarrow$ quantized hydrodynamics.

Correlation functions in single component case

Single particle density matrix:
$\langle \psi_B(x)\psi_B^\dagger(x')\rangle \sim |x-x'|^{-1/(2K)}$,
Density density correlations:
$\langle \rho(x)\rho(x')\rangle \sim \rho_0^2 + C/|x-x|^2 + \cos(2\pi\rho_0(x-x'))/|x-x'|^{2K}$.
$\Rightarrow$ determined by the Luttinger exponent K.

- uK related to sensitvity of ground state energy to change of boundary conditions = Kohn stiffness.
- u/K related to sensitvity of ground state energy to change of particle number = compressibility.

Multicomponent case

1. Bose-Bose, Bose-Fermi or Fermi-Fermi mixture in 1D.
2. Effective Hamiltonian for the collective modes ?
3. Parameters of the Hamiltonian ?
4. correlation functions ?

Second Quantized Hamiltonian

$$
\begin{aligned}
H &= \sum_{j=1}^{N} -\frac{\hbar^2}{2M_j} \int dx \psi_j^\dagger(x) \partial_x^2 \psi_j(x') \\
&+ \frac{1}{2} \int dx dx' \sum_{1\le i,j \le N} V_{ij}(x-x') \rho_i(x) \rho_j(x')
\end{aligned}
$$

with:

$$
\begin{aligned}
[\psi_j^\dagger(x), \psi_j(x')]_\pm &= \delta(x-x') \\
\rho_j(x) &= \psi_j^\dagger(x) \psi_j(x)
\end{aligned}
$$

(+=fermions, -=bosons)
V_{ij} are short ranged.

Phenomenological bosonization

(Haldane, 1980)

$$
\begin{aligned}
\pi \Pi_j(x) &= \partial_x \theta_j ; [\phi_j(x), \Pi_k(x')] = i \delta_{jk} \delta(x - x') \, \mathrm{(CCR)} \\
\rho_j(x) &= \rho_j^{(0)} - \frac{1}{\pi} \partial_x \phi_j + \sum_{m \geq 1} \frac{A_m^{(j)}}{\pi a} \cos(2m\phi_j(x) - 2\pi m \rho_j^{(0)} x) \\
\psi_j(x) &= e^{i\theta_j(x)} \sum_p \frac{B_p^{(j)}}{\sqrt{2\pi a}} cos(p\phi_j(x) - \pi p \rho_j^{(0)} x)
\end{aligned}
$$

(p odd for fermions, even for bosons).

Symmetries

- global gauge invariance: $\theta_j \to \theta_j + \alpha_j$
- translational invariance: $\phi_j \to \phi_j + \pi \rho_j^{(0)} a$
- Parity: $\phi_j(x) \to -\phi_j(-x)$ and $\theta_j(x) \to \theta_j(-x)$
- Time reversal: $\phi_j(x) \to \phi_j(x)$, $\theta_j(x) \to -\theta_j(x)$ and $i \to -i$

Phenomenological Hamiltonian

Incommensurate case. Most general quadratic Hamiltonian:

$$H = \sum_{j,k} \int \frac{dx}{2\pi} \left[\pi^2 M_{jk} \Pi_j \Pi_k + N_{jk} \partial_x \phi_j \partial_x \phi_k\right],$$

compatible with symmetries.
Valid for low energy.

Microscopic to phenomenological (I)

Consider a system with finite size L.
Adding particles: $\rho_j \to \rho_j + q_j/L$.

$$\delta E_0 = \sum_{j=1}^{N} \frac{\partial E_0}{\partial N_j} q_j + \frac{1}{2} \sum_{1\leq j,k \leq N} \frac{\partial^2 E_0}{\partial N_j \partial N_k} q_j q_k$$

In phenomenological Hamiltonian: $\partial_x \phi_j \to \partial_x \phi_j - \pi q_j/L$.

$$\Rightarrow N_{jk} = \frac{L}{\pi} \frac{\partial^2 E_0}{\partial N_j \partial N_k}.$$

$\Rightarrow N_{jk}$: inverse compressibilities

Microscopic to phenomenological (II)

Consider a system with finite size L.
Modify boundary conditions: $\psi_j(L) = e^{i\varphi_j}\psi_j(0)$.

$$\delta E_0 = \frac{1}{2}\sum_{j,k} \frac{\partial^2 E_0}{\partial\varphi_j \partial\varphi_k} q_j q_k$$

In phenomenological Hamiltonian: $\partial_x \theta_j \to \partial_x \theta_j + \varphi_j/L$.
Identification of ground state energy changes:

$$\Rightarrow M_{jk} = \pi L \frac{\partial^2 E_0}{\partial\varphi_j \partial\varphi_k},$$

$\Rightarrow$ *M*: Kohn's stiffness

Galilean invariant case

Particle current: $J_j(x) = \partial_t \phi_j(x,t)/\pi$
E. O. M. $\Rightarrow J_j = \sum_{k=1}^N M_{jk} \partial_x \theta_k/\pi$
Galilean boost: $\theta_j(r) \rightarrow \theta_j(r) + M_j v r \Rightarrow \langle J_j \rangle = \frac{1}{\pi} \sum_{k=1}^N M_{jk} M_k v$
Under galilean boost: $\langle J_j \rangle = \rho_j^{(0)} v$

$$\Rightarrow \pi \rho_j^{(0)} = \sum_{k=1}^N M_{jk} M_j$$

Diagonalization of bosonized Hamiltonian

$$
\begin{aligned}
\Delta_1 &= {}^tR_1 M R_1 \\
\Delta_2 &= {}^tR_2 \Delta_1^{1/2}\, {}^tR_1 N R_1 \Delta_1^{1/2} R_2 \\
\theta &= R_1 \Delta_1^{-1/2} R_2 \Delta_2^{1/4} \vartheta = Q\vartheta \\
\phi &= R_1 \Delta_1^{1/2} R_2 \Delta_2^{-1/4} \varphi = P\varphi \\
{}^tP &= Q^{-1} \\
H &= \int \frac{dx}{2\pi} \left[{}^t(\partial_x \vartheta) \Delta_2^{1/2} \partial_x \vartheta + {}^t(\partial_x \varphi) \Delta_2^{1/2} \partial_x \varphi \right]
\end{aligned}
$$

$\Delta_2^{1/2}$=diagonal matrix. Its eigenvalues are the velocities of the modes. (Emery and Muttalib, 1986)

Equal time correlations, zero temperature

$$\langle e^{i\sum_a \lambda_a \phi_a(x,0)} e^{-i\sum_a \lambda_a \phi_a(0,0)} \rangle = \left(\frac{\alpha}{\sqrt{x^2+\alpha^2}}\right)^{\frac{1}{2}{}^t\lambda (MN)^{-1/2} M \lambda},$$

$$\langle e^{i\sum_a \lambda_a \theta_a(x,0)} e^{-i\sum_a \lambda_a \theta_a(0,0)} \rangle = \left(\frac{\alpha}{\sqrt{x^2+\alpha^2}}\right)^{\frac{1}{2}{}^t\lambda (NM)^{-1/2} N \lambda},$$

For single component, respective exponents $\lambda^2/(2K)$ and $\lambda^2 K/2$.
$K \leftrightarrow 1/K$ duality becomes $M \leftrightarrow N$ duality.

Equal time correlations, positive temperature

$$\langle e^{i\sum_a \lambda_a \phi_a(x,0)} e^{-i\sum_a \lambda_a \phi_a(0,0)}\rangle \sim e^{-\frac{\pi T}{2} {}^t\lambda N^{-1}\lambda |x|},$$
$$\langle e^{i\sum_a \lambda_a \theta_a(x,0)} e^{-i\sum_a \lambda_a \theta_a(0,0)}\rangle = e^{-\frac{\pi T}{2} {}^t\lambda M^{-1}\lambda |x|},$$

Thermal lengths:

$$\xi_\phi(\lambda) = 2/(\pi T {}^t\lambda(N)^{-1}\lambda)$$
$$\xi_\theta(\lambda) = 2/(\pi T {}^t\lambda(M)^{-1}\lambda)$$

Time-dependent correlations

Density-density correlations and Green's functions for the bosons: $\rightarrow$ A. Iucci, G. Fiete, T. Giamarchi Phys. Rev. B (2007).

Retardated Green's function for fermions

$$\begin{aligned} G_a(x,t) &= -i\theta(t)\langle\{\psi(x,t),\psi^\dagger(0,0)\}\rangle \\ A_a(q,\omega) &= \int dx \int dt G_a(x,t) e^{-i(qx-\omega t)} \end{aligned}$$

$A_a =$ spectral function. Measured in ARPES for $\omega > 0$.

Relevant Correlation functions

$$\langle e^{i(\theta_a-\phi_a)(x,t)} e^{-i(\theta_a-\phi_a)(0,0)} \rangle =$$
$$= e^{\langle (\theta_a(x,t)-\phi_a(x,t))(\theta_a(0,0)-\phi_a(0,0))\rangle_{conn.}}$$
$$\langle (\theta_a(x,t)-\phi_a(x,t))(\theta_a(0,0)-\phi_a(0,0))\rangle_{conn.}$$
$$= \sum_b \nu_b \ln\left(\frac{\alpha}{\alpha+i(v_b t - x)}\right) + \nu'_b \ln\left(\frac{\alpha}{\alpha+i(v_b t + x)}\right)$$
$$\nu_b = (P_{ab}+Q_{ab})^2/4$$
$$\nu'_b = (P_{ab}-Q_{ab})^2/4$$

Two-species case

$T = 0$ spectral functions

$$
\begin{aligned}
A(k_F + q, \omega) &= I(q,\omega) + I(-q,-\omega) \\
I(q,\omega) &= \frac{C\alpha^{\nu_1+\nu_2+\nu_1'+\nu_2'}}{2\pi} \\
&\times \int \frac{e^{i(\omega t - qx)}}{(\alpha + i(v_1 t - x))^{\nu_1}(\alpha + i(v_2 t - x))^{\nu_2}} \\
&\times \frac{dxdt}{(\alpha + i(v_1 t + x))^{\nu_1'}(\alpha + i(v_2 t + x))^{\nu_2'}}
\end{aligned}
$$

Feynman identity

$$
\frac{1}{A_1^{\nu_1} A_2^{\nu_2}} = \frac{\Gamma(\nu_1+\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_0^1 du \frac{u^{\nu_1-1}(1-u)^{\nu_2-1}}{(A_1 u + A_2(1-u))^{\nu_1+\nu_2}}
$$

$$I(q,\omega) = \frac{\pi^2}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_1')\Gamma(\nu_2')} \int_0^1 du_1 u_1^{\nu_1-1}(1-u_1)^{\nu_2-1}$$
$$\times(\omega - v(u_1)q)^{\nu_2'+\nu_1'-1} \int_0^1 du_2 u_2^{\nu_1'-1}(1-u_2)^{\nu_2'-1}$$
$$\times(\omega + v(u_2)q)^{\nu_2+\nu_1-1} \frac{\Theta(\omega - v(u_1)q)\Theta(\omega + v(u_2)q)}{(v(u_1)+v(u_2))^{\nu_2'+\nu_1'\nu_2+\nu_1-1}}$$
$$v(u) = uv_2 + (1-u)v_1$$

Threshold

$I(q>0, \omega < v_2 q) = 0 \Rightarrow A(q>0, |\omega| < v_2 q) = 0$

Power law singularities of the spectral function

V. Meden, K. Schoenhammer Phys. Rev. B **46**, (1992) J. Voit Phys. Rev. B**47**, (1992)

$$
\begin{aligned}
A(q,\omega) &\sim (\omega - v_1 q)^{\nu_2+\nu_1'+\nu_2'-1} (\omega \simeq v_1 q) \\
A(q,\omega) &\sim (\omega - v_2 q)^{\nu_1+\nu_1'+\nu_2'-1} (\omega \simeq v_2 q) \\
A(q,\omega) &\sim (-\omega - v_2 q)^{\nu_1'+\nu_1+\nu_2-1} (\omega \simeq -v_2 q) \\
A(q,\omega) &\sim C + C'|\omega + v_1 q|^{\nu_2'+\nu_1+\nu_2-1} (\omega \simeq -v_1 q)
\end{aligned}
$$

$0 < v_2 q < \omega < v_1 q$

$$
\begin{aligned}
I(q,\omega) &= \frac{\pi^2}{\Gamma(\nu_2)\Gamma(\nu'_1)\Gamma(\nu'_2)\Gamma(\nu_1+\nu'_1+\nu'_2)} \\
&\times \frac{(\omega - v_2 q)^{\nu'_2+2\nu'_1-1/2}(v_1 q-\omega)^{2\nu'_2+\nu'_1-1/2}}{(v_1-v_2)^{2\nu'_1+2\nu'_2}(\omega+v_2 q)^{\nu'_2}(\omega+v_2 q)^{\nu'_1}} \\
&\times \int_0^1 t^{\nu'_2-1}(1-t)^{\nu'_1-1} \\
&{}_2F_1\left(2(\nu'_1+\nu'_2),\nu'_1+\nu'_2;\nu'_2+2\nu'_1+1/2;\right. \\
&1+\frac{\omega-v_1 q}{(v_1-v_2)q} \\
&\left.\times\frac{(\omega+v_2 q)(v_1+v_2)+(v_1-v_2)(v_2 q-\omega)t}{(\omega+v_2 q)(\omega+v_1 q)}\right) dt
\end{aligned}
$$

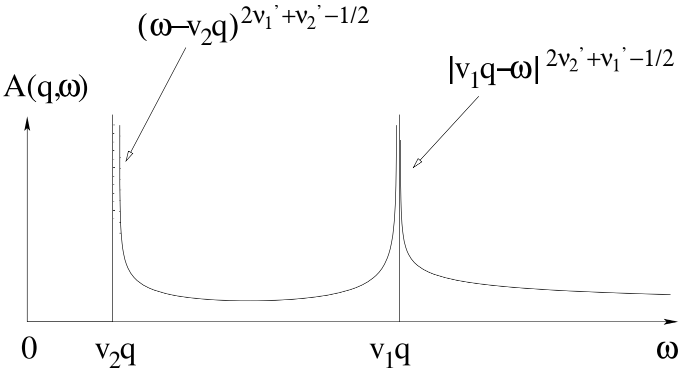
${}_2F_1$ is Gauss' hypergeometric function.

$\omega > v_1 q$

$$I(q,\omega) = \frac{\pi\alpha^{\nu_1+\nu_2+\nu_1'+\nu_2'-1}(\omega - v_2 q)^{\nu_1+\nu_1'+\nu_2'-1}(\omega + v_2 q)^{\nu_1+\nu_2+\nu_1'-1}}{2\Gamma(\nu_1+\nu_2)\Gamma(\nu_1'+\nu_2')(\omega - v_1 q)^{\nu_1}(\omega + v_1 q)^{\nu_1'}}$$
$$\times F_2\left(\nu_1+\nu_2+\nu_1'+\nu_2'-1; \nu_1, \nu_1'; \nu_1+\nu_2, \nu_1'+\nu_2'; \frac{v_2-v_1}{2v_2}\frac{\omega+v_2 q}{\omega - v_1 q}\right.$$
$$\left., \frac{v_2-v_1}{2v_2}\frac{\omega - v_2 q}{\omega + v_1 q}\right),$$

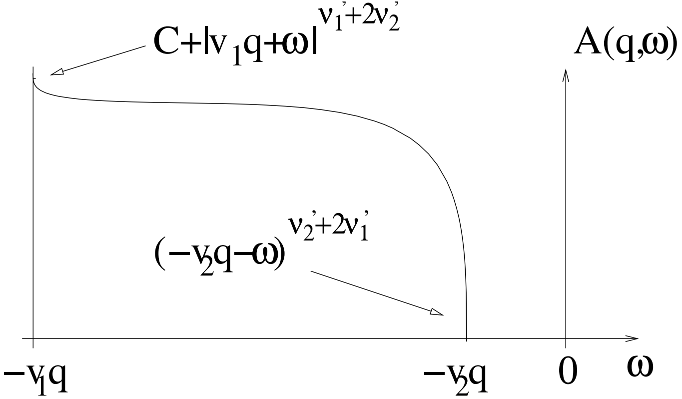
F_2 is an Appell hypergeometric function.
For the bosonic case, Iucci et al. (2007) had F_1.

spectral function for $\omega > 0$: peaks



$(\nu_a = \nu'_a + 1/2)$

spectral function for $-v_1 q < \omega < 0$: cusps



$(\nu_a = \nu_a' + 1/2)$

Conclusions

1. Derivation of a phenomenological Hamiltonian
2. Relations coming from Galilean invariance
3. Expression of correlation/spectral functions

Reference: E. Orignac M. Tsuchiizu and Y. Suzumura Phys. Rev. A **81**, 053626 (2010).

Perspectives

1. Application to integrable models (Demler-Imambekov, 2005)
2. Spectral functions in multicomponent case (> 2 species)
3. Commensurate case