Spreading and thermalization in disordered nonlinear chains

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- Anderson localization 1958 - 2008: Introduction, 50 years after
- Discrete Anderson nonlinear Schrödinger equation (DANSE) ($d = 1, 2$)
- Dynamical thermalization of nonlinear disordered lattices: anti-FPU
- Fast Arnold diffusion: unknown result of Chirikov and Vecheslavov (1997)

Anderson localization: introduction & perspectives

from the talk of P.W. Anderson at Newton Institute, July 21, 2008
see http://www.newton.ac.uk/programmes/MPA/seminars/072117001.html

Perspectives: a) localization in new type of systems; b) effects of interactions.
3d-Dynamical de-localization of atomic waves

quantum chaos in kicked rotator => Chirikov localization in momentum space

=> dynamical analog of 3d Anderson transition

\[ H = \frac{p^2}{2} + K \cos x \left[ 1 + \epsilon \cos(\omega_2 t) \cos(\omega_3 t) \right] \sum_m \delta(t - m), \quad \hbar_{\text{eff}} = 2.89 \]

Nonlinearity and Anderson localization: estimates

\[ i\hbar \frac{\partial \psi_n}{\partial t} = E_n \psi_n + \beta |\psi_n|^2 \psi_n + V(\psi_{n+1} + \psi_{n-1}); \left[-W/2 < E_n < W/2\right] \]

localization length \( l \approx 96(V/W)^2 \) (1D); \( l \approx (V/W)^2 \) (2D) Amplitudes \( C \) in the linear eigenbasis are described by the equation

\[ i \frac{\partial C_m}{\partial t} = \epsilon_m C_m + \beta \sum_{m_1 m_2 m_3} U_{mm_1 m_2 m_3} C_{m_1} C_{m_2}^* C_{m_3} \]

the transition matrix elements are \( U_{mm_1 m_2 m_3} = \sum_n Q_{nm}^{-1} Q_{nm_1} Q_{nm_2}^* Q_{nm_3} \sim 1/l^{3d}/2 \). There are about \( l^{3d} \) random terms in the sum with \( U \sim l^{-3d}/2 \) so that we have \( idC/dt \sim \beta C^3 \). We assume that the probability is distributed over \( \Delta n > l^d \) states of the lattice basis. Then from the normalization condition we have \( C_m \sim 1/(\Delta n)^{1/2} \) and the transition rate to new non-populated states in the basis \( m \) is \( \Gamma \sim l^{2} |C|^6 \sim \beta^2/(\Delta n)^3 \).

Due to localization these transitions take place on a size \( l \) and hence the diffusion rate in the distance \( \Delta R \sim (\Delta n)^{1/d} \) of \( d \) – dimensional \( m \) – space is \( d(\Delta R)^2/dt \sim l^2 \Gamma \sim \beta^2 l^2/(\Delta n)^3 \sim \beta^2 l^2/(\Delta R)^{3d} \). At large time scales \( \Delta R \sim R \) and we obtain

\[ \Delta n \sim R^d \sim (\beta l)^{2d/(3d+2)} t^{d/(3d+2)}; (\Delta n)^2 \propto t^\alpha; \alpha = 2/(3d+2) \]

Chaos criterion:

\[ S = \delta \omega/\Delta \omega \sim \beta > \beta_c \sim 1 \]

there \( \delta \omega \sim \beta |\psi_n|^2 \sim \beta/\Delta n \) is nonlinear frequency shift

and \( \Delta \omega \sim 1/\Delta n \) is spacing between exit states eigenmodes

DLS PRL 70, 1787 (1993) \( (d=1) \);

I.García-Mata, DLS PRE 79, 026205 (2009) \( (d \geq 1) \)
Nonlinearity and Anderson localization (1D)

\[ W/V = 2, 4, \beta = 0, 1; \sigma = (\Delta n)^2 \propto t^\alpha; \]
\[ \alpha = 2/5 \text{ (theory) 0.34, 0.31 numerics} \]

\[ i\hbar \frac{\partial \psi_n}{\partial t} = E_n \psi_n + \beta |\psi_n|^2 \psi_n + V(\psi_{n+1} + \psi_{n-1}); \quad [-W/2 < E_n < W/2] \]

A.S. Pikovsky, DLS PRL 100, 094101 (2008)
Nonlinearity and Anderson localization (2D)

\[ W/V = 10, 15, \beta = 0, 1; \alpha_2 = 0.236, 0.229 \pm 0.003 \text{ (theory 0.25)} \]

\[ \nu = 0.282, 0.247 \pm 0.005 \text{ (theory 0.25); } \xi \text{ is participation ratio} \]

\[ i\hbar \frac{\partial \psi_n}{\partial t} = E_n \psi_n + \beta |\psi_n|^2 \psi_n + V(\psi_{n+1} + \psi_{n-1}) \]

Nonlinearity and Anderson localization (2D)

$W = 10; \beta = 0$ (left), $1$ (right);
$t = 10^4$ (bottom), $10^6$ (middle),
projecton on $x$–axis (top);
256 × 256 lattice

[also: kicked nonlinear rotator model (1d)]

Delocalization on disordered Stark ladder

Static field \( f \) along Stark ladder (\( W = 4 \)): statistical entanglement

Left: \( f = 0, 0.25, 0.5, \alpha = 0.30, 0.26, 0.24, \beta = 1; 0 \) top to bottom; inset IPR at \( f = 0.5 \);

Right: probability distribution at \( f = 0.5, t = 10^2, 10^4, 10^6, 10^8, \beta = 0; 1 \) (top/bottom)

I. García-Mata, DLS EPJB 71, 121 (2009)
Dynamical thermalization in DANSE (1D)

starting from Fermi-Pasta-Ulam problem (1955):
regular lattice, delocalized linear modes $\rightarrow$ disorder localized modes

Gibbs distribution with temperature $T$ for localized linear modes, $\rho_m = |C_m|^2$:
- entropy $S = -\sum_m \rho_m \ln \rho_m$, $\rho_m = Z^{-1} \exp(-\epsilon_m/T)$, $Z = \sum_m \exp(-\epsilon_m/T)$,
- $E = T^2 \partial \ln Z / \partial T$, $S = E / T + \ln Z$. $\langle \ln Z \rangle \approx \ln N + \ln \sinh(\Delta/T) - \ln(\Delta/T)$, $\Delta \approx 3$

Dynamical thermalization in DANSE (1D)

weaker and stronger nonlinearity $\beta$

$N = 64, W = 4, \beta = 0.5(left), 2(right), t = 10^6$, initial state: one linear eigenmode

Gibbs distribution with temperature $T$ for localized linear modes, $\rho_m = |C_m|^2$:

entropy $S = -\sum_m \rho_m \ln \rho_m, \quad \rho_m = Z^{-1} \exp(-\epsilon_m/T), \quad Z = \sum_m \exp(-\epsilon_m/T), \quad E = T^2 \partial \ln Z / \partial T, \quad S = E / T + \ln Z . \quad \langle \ln Z \rangle \approx \ln N + \ln \sinh(\Delta / T) - \ln(\Delta / T), \quad \Delta \approx 3$

Dynamical thermalization in DANSE (1D)

\( N = 32, W = 4, \beta = 1, t = 10^6 \), initial state: linear eigenmode \( m' \), averaged over 8 disorder realisations

Gibbs distribution: time, disorder averaged \( \rho_m \) in mode \( m \) (y-axis) for initial eigenmode \( m' \) (x-axis); left: numerics, right: Gibbs theory

Dynamical thermalization in DANSE (1D)

Fraction of thermalized states: $N = 16$ (circles), 32 (curve), 64(+) ; $W = 4$, $t = 10^6$, (diamonds $N = 32$, $t = 10^7$)

Possible experimental tests & applications

- BEC in disordered potential (Aspect, Inguscio)
- kicked rotator with BEC (Phillips)
- nonlinear wave propagation in disordered media (Segev, Silberberg)
- lasing in random media (Cao)
- energy propagation in complex molecular chains (proteins, Fermi-Pasta-Ulam problem)
- NONLINEAR SPIN-GLASS ?

OTHER GROUPS:
S.Aubry et al. PRL **100**, 084103 (2008)
A.Dhar et al. PRL **100**, 134301 (2008)
see also the participant list of the NLSE Workshop at the Lewiner Institute, Technion, June 2008
Instead of conclusion: Fast Arnold Diffusion

- Outstanding result unknown to the community
- Diffusion $D$ inside chaotic layers in nonlinear systems with many degrees of freedom:
  \[ \ln D \propto \epsilon^\mu \]
  at weak perturbation $\epsilon$ with $\mu \approx 6.6$ up to exponentially small $\epsilon$ and $D \sim 10^{-50}$